Review of Matrix Algebra

Steven Vukazich
San Jose State University

Notation

Vector

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

Matrix

$$[A] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \\ A_{14} & A_{24} & A_{34} \end{bmatrix}$$

It is sometimes useful to show the dimension (number of rows by number of columns) of a matrix or vector below the matrix or vector

$[A]$ $\{d\}$

4 x 3 4 x 1

number of columns

number of rows

Transpose

Vector

$$\underset{3 \times 1}{\{d\}} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \qquad \underset{1 \times 3}{\{d\}^T} = \{d_1, d_2, d_3\}$$

Matrix

$$\underset{3 \times 2}{[A]} = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 8 \end{bmatrix} \qquad \underset{2 \times 3}{[A]^T} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 8 \end{bmatrix}$$

Symmetric matrix

$$[A] = \begin{bmatrix} 4 & 1 & 0 & 7 \\ 1 & 6 & 3 & 8 \\ 0 & 3 & 9 & 2 \\ 7 & 8 & 2 & 11 \end{bmatrix}$$

Diagonal matrix

$$[D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Vector Scalar (Dot) Product

$$\underset{3 \times 1}{\{d\}} = \begin{Bmatrix} 2 \\ 1 \\ 4 \end{Bmatrix}$$

$$\underset{3 \times 1}{\{b\}} = \begin{Bmatrix} 5 \\ 2 \\ 6 \end{Bmatrix}$$

$$\boldsymbol{d} \cdot \boldsymbol{b} = \underset{1 \times 3}{\{d\}^T} \underset{3 \times 1}{\{b\}} = \{2, 1, 4\} \begin{Bmatrix} 5 \\ 2 \\ 6 \end{Bmatrix}$$

$$= (2)(5) + (1)(2) + (4)(6)$$

$$= 10 + 2 + 24 = \mathbf{36}$$

Multiplication of a Matrix and a Vector

$$[A] = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$
3 x 2

$$\{b\} = \begin{Bmatrix} 5 \\ 2 \end{Bmatrix}$$
2 x 1

$$[A]\{b\} = \{c\}$$
3 x 2   2 x 1   3 x 1

must be the same

$$\begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 8 \end{bmatrix} \begin{Bmatrix} 5 \\ 2 \end{Bmatrix}$$

$$= (2)(5) + (1)(2) = 12$$
$$(3)(5) + (6)(2) = 27$$
$$(4)(5) + (8)(2) = 36$$

$$\{c\} = \begin{Bmatrix} 12 \\ 27 \\ 36 \end{Bmatrix}$$
3 x 1

Identity matrix

$$\underset{4 \times 4}{[I]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix is a square diagonal matrix with ones on the diagonal. It is the matrix analog of the number 1

The identity matrix has the following properties:

$$\underset{n \times n}{[I]}\underset{n \times 1}{\{b\}} = \underset{n \times 1}{\{b\}} \qquad [I]^T = [I]$$

$$\underset{n \times n}{[I]}\underset{n \times n}{[A]} = \underset{n \times n}{[A]} \qquad \underset{n \times n}{[A]}\underset{n \times n}{[I]} = \underset{n \times n}{[A]}$$

Determinant of a 2 x 2 matrix

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The determinant of a 2 x 2 matrix is defined as:

$$\det[A] = A_{11}A_{22} - A_{12}A_{21}$$

The determinant of all square matrices is defined and can be derived from the determinant of a 2 x 2 matrix.

Singular Matrix

A square matrix is singular if:

$$\det[A] = 0$$

A square matrix is nonsingular if:

$$\det[A] \neq 0$$

Inverse of a Matrix

All nonsingular square matrices have an inverse that satisfies:

$$[A]^{-1}[A] = [I]$$

The inverse of a 2 x 2 matrix is:

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$[A]^{-1} = \frac{1}{\det[A]}\begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

$$\det[A] = A_{11}A_{22} - A_{12}A_{21}$$

The inverse of all larger square nonsingular matrices is defined and can be found using techniques beyond the scope of this review

System of Equations in Matrix Form

A system of n equations n unknowns can be represented in matrix form as:

$$[A]\{x\} = \{b\}$$

Where:
$\{x\}$ is the vector of unknowns;
$[A]$is the matrix of known coefficients;
$\{b\}$ is the vector of known data

For example, the system of 3 equations and 3 unknowns can be represented in matrix form as:

$$\begin{aligned} 5x_1 + 6x_2 + x_3 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 5 \\ x_2 + 6x_3 &= 7 \end{aligned}$$

$$\begin{bmatrix} 5 & 6 & 1 \\ 4 & 9 & 2 \\ 0 & 1 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 7 \end{Bmatrix}$$

Solution of a System of Linear Equations

A system of n equations n unknowns has a unique solution if the coefficient matrix is nonsingular ($\det[A] \neq 0$)

In theory, the solution can be found by finding the inverse of $[A]$ and pre-multiplying the both sides of the system of equations by $[A]^{-1}$

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{b\}$$

$$\{x\} = [A]^{-1}\{b\}$$

Note that in practice, there are more efficient methods of solving a system of linear equations.