

# Review of Matrix Algebra

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# Notation

## Vector

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

It is sometimes useful to show the dimension (number of rows by number of columns) of a matrix or vector below the matrix or vector

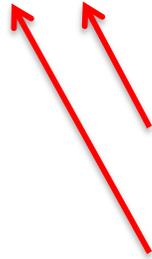
## Matrix

$$[A] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \\ A_{14} & A_{24} & A_{34} \end{bmatrix}$$

$$[A] \quad \{d\}$$

4 x 3      4 x 1

number of columns  
number of rows



# Transpose

Vector

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

3 x 1

$$\{d\}^T = \{d_1, d_2, d_3\}$$

1 x 3

Matrix

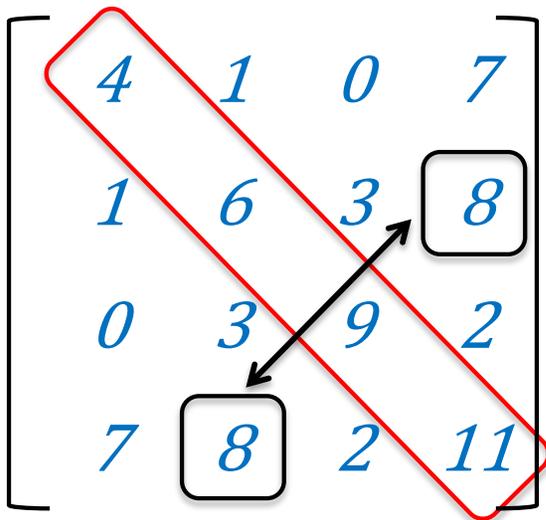
$$[A] = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

3 x 2

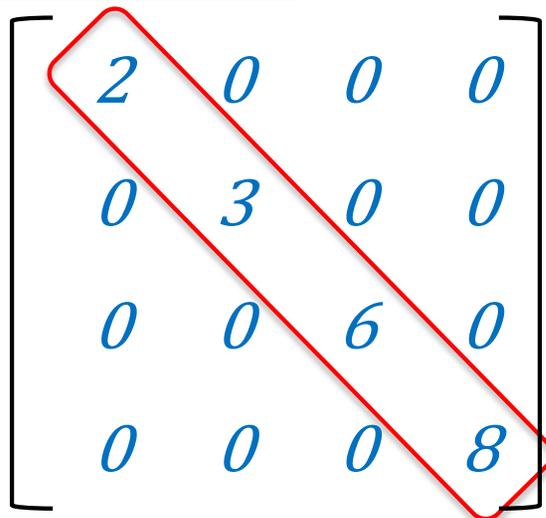
$$[A]^T = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 6 & 8 \end{bmatrix}$$

2 x 3

## Symmetric matrix

$$[A] = \begin{bmatrix} 4 & 1 & 0 & 7 \\ 1 & 6 & 3 & 8 \\ 0 & 3 & 9 & 2 \\ 7 & 8 & 2 & 11 \end{bmatrix}$$
A 4x4 matrix is shown with a red diagonal line. The elements 8 in the second row, fourth column and the fourth row, second column are boxed. A double-headed arrow connects these two boxed elements, illustrating the symmetry of the matrix.

## Diagonal matrix

$$[D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$
A 4x4 diagonal matrix is shown with a red diagonal line. The diagonal elements are 2, 3, 6, and 8. All off-diagonal elements are 0.

## Vector Scalar (Dot) Product

$$\begin{matrix} \{d\} \\ 3 \times 1 \end{matrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \quad \mathbf{d} \cdot \mathbf{b} = \underbrace{\{d\}^T}_{1 \times 3} \underbrace{\{b\}}_{3 \times 1} = \{2, 1, 4\} \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{matrix} \{b\} \\ 3 \times 1 \end{matrix} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} \quad = (2)(5) + (1)(2) + (4)(6)$$

$$= 10 + 2 + 24 = \mathbf{36}$$

# Multiplication of a Matrix and a Vector

$$[A] = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

$3 \times 2$

$$\{b\} = \begin{Bmatrix} 5 \\ 2 \end{Bmatrix}$$

$2 \times 1$

$$[A]\{b\} = \{c\}$$

$3 \times 2 \quad 2 \times 1 \quad 3 \times 1$

must be the same

$$\begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

$$\begin{Bmatrix} 5 \\ 2 \end{Bmatrix}$$

$$= (2)(5) + (1)(2) = 12$$

$$(3)(5) + (6)(2) = 27$$

$$(4)(5) + (8)(2) = 36$$

$$\{c\} = \begin{Bmatrix} 12 \\ 27 \\ 36 \end{Bmatrix}$$

$3 \times 1$

## Identity matrix

$$[I]_{4 \times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix is a square diagonal matrix with ones on the diagonal. It is the matrix analog of the number 1

The identity matrix has the following properties:

$$\begin{matrix} [I] \{b\} = \{b\} \\ n \times n & n \times 1 & n \times 1 \end{matrix}$$

$$[I]^T = [I]$$

$$\begin{matrix} [I][A] = [A] \\ n \times n & n \times n & n \times n \end{matrix}$$

$$\begin{matrix} [A][I] = [A] \\ n \times n & n \times n & n \times n \end{matrix}$$

## Determinant of a 2 x 2 matrix

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The determinant of a 2 x 2 matrix is defined as:

$$\det[A] = A_{11}A_{22} - A_{12}A_{21}$$

The determinant of all square matrices is defined and can be derived from the determinant of a 2 x 2 matrix.

## Singular Matrix

A square matrix is **singular** if:

$$\det[A] = 0$$

A square matrix is **nonsingular** if:

$$\det[A] \neq 0$$

## Inverse of a Matrix

All **nonsingular** square matrices have an inverse that satisfies:

$$[A]^{-1}[A] = [I]$$

The inverse of a 2 x 2 matrix is:

$$[A]^{-1} = \frac{1}{\det[A]} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\det[A] = A_{11}A_{22} - A_{12}A_{21}$$

The inverse of all larger square nonsingular matrices is defined and can be found using techniques beyond the scope of this review

# System of Equations in Matrix Form

A system of  $n$  equations  $n$  unknowns can be represented in matrix form as:

$$[A]\{x\} = \{b\}$$

Where:

$\{x\}$  is the vector of unknowns;

$[A]$  is the matrix of known coefficients;

$\{b\}$  is the vector of known data

For example, the system of 3 equations and 3 unknowns can be represented in matrix form as:

$$\begin{aligned}5x_1 + 6x_2 + x_3 &= 2 \\4x_1 + 9x_2 + 2x_3 &= 5 \\x_2 + 6x_3 &= 7\end{aligned}$$

$$\begin{bmatrix} 5 & 6 & 1 \\ 4 & 9 & 2 \\ 0 & 1 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 7 \end{Bmatrix}$$

# Solution of a System of Linear Equations

A system of  $n$  equations  $n$  unknowns has a **unique solution** if the coefficient matrix is **nonsingular** ( $\det[A] \neq 0$ )

In theory, the solution can be found by finding the inverse of  $[A]$  and pre-multiplying the both sides of the system of equations by  $[A]^{-1}$

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{b\}$$

$$\{x\} = [A]^{-1}\{b\}$$

Note that in practice, there are more efficient methods of solving a system of linear equations.