# Method of Virtual Work for Beams and Frames

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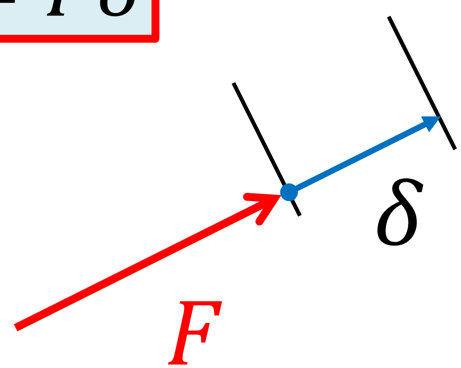
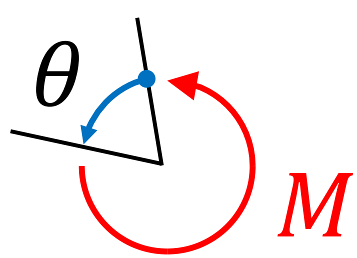
# Work Done by Force/Moment

$$W = F\delta$$




Work is done by a force acting through and in-line displacement

$$W = M\theta$$




Work is done by a moment acting through and in-line rotation

# Recall the General Form of the Principle of Virtual Work

Real Deformation

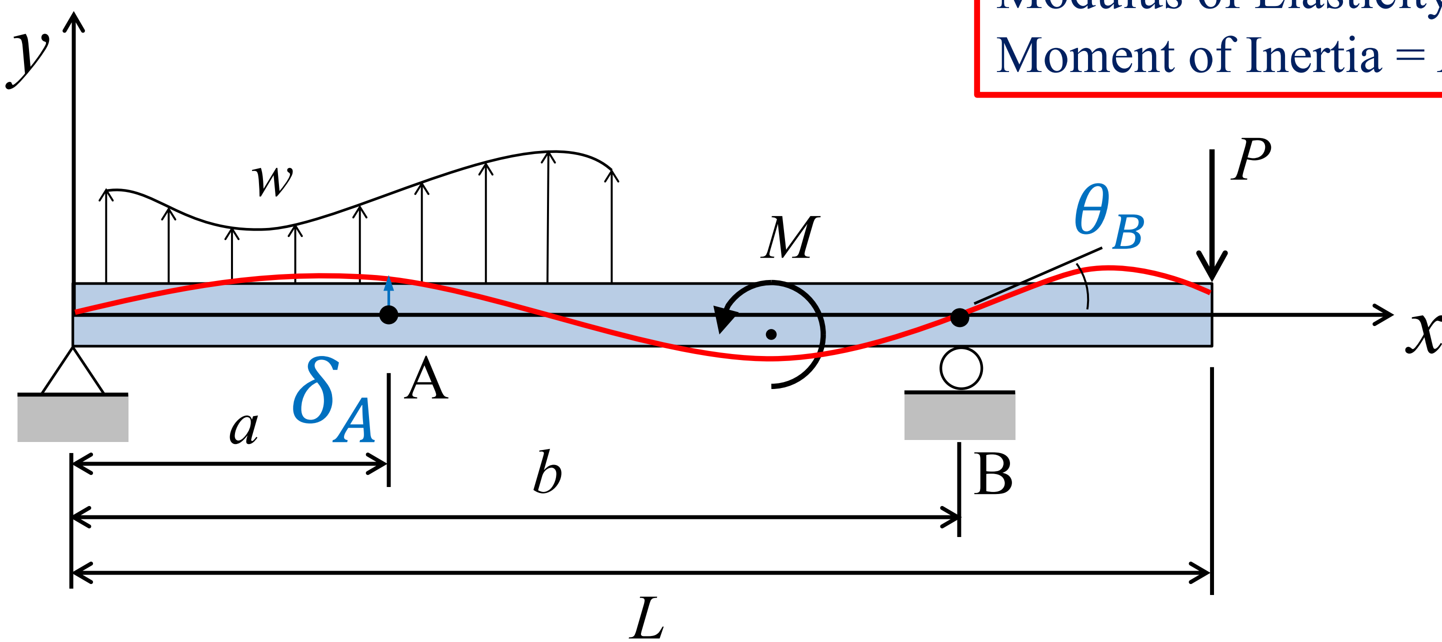
$$Q\,\delta_P + \sum R_Q\,\delta_s = \iint F_Q\,dL_P$$

Virtual Loads

External Work due to Support Settlements

# Consider a Beam Subjected To General Loading

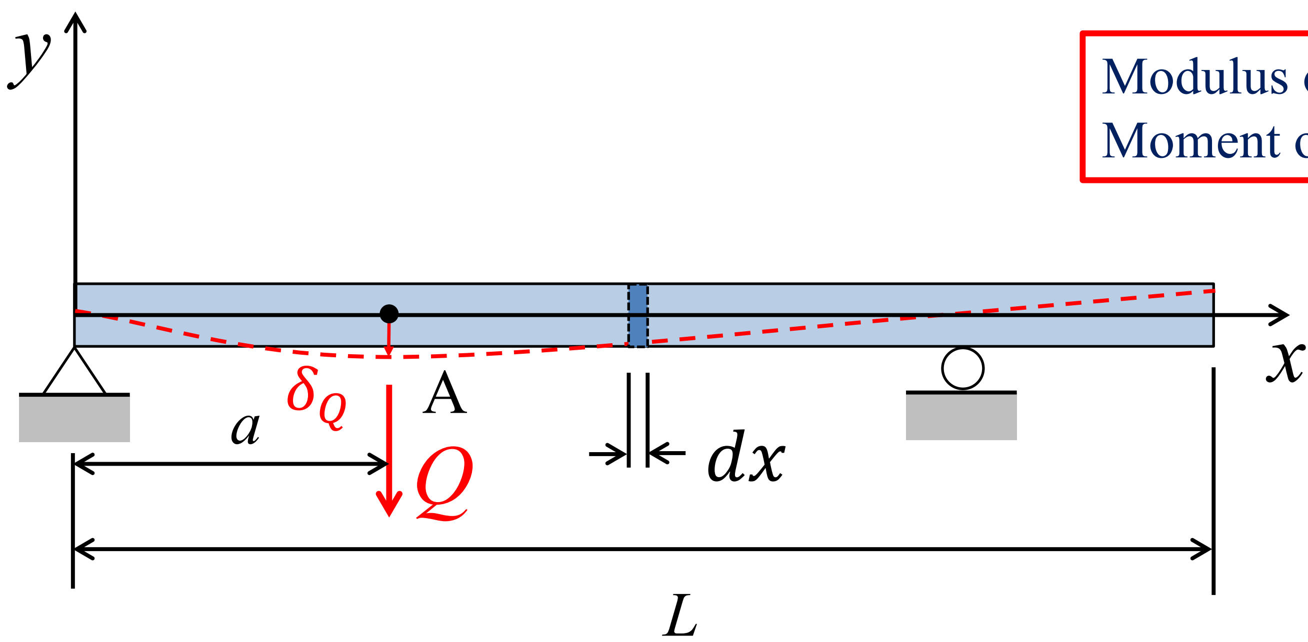
Modulus of Elasticity = $E$
Moment of Inertia = $I$

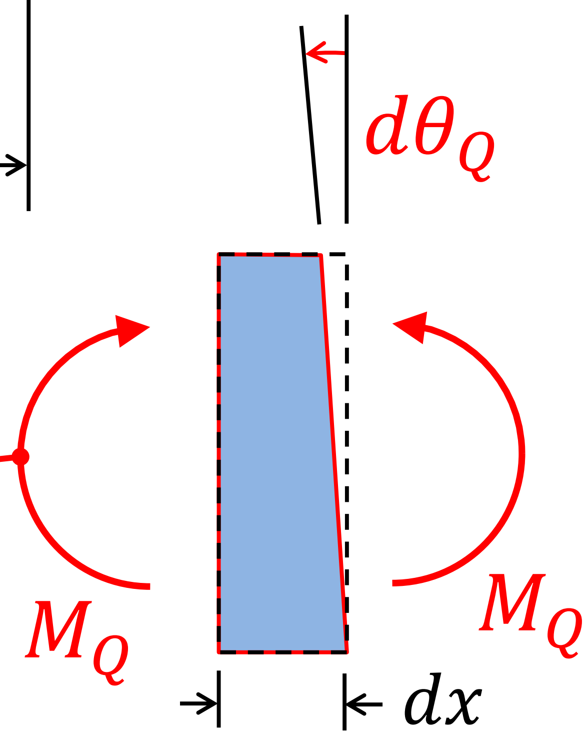



We want to find the deflection at point A and the slope at point B due to the applied loads

# Apply a Virtual Force to Measure the Deflection at A

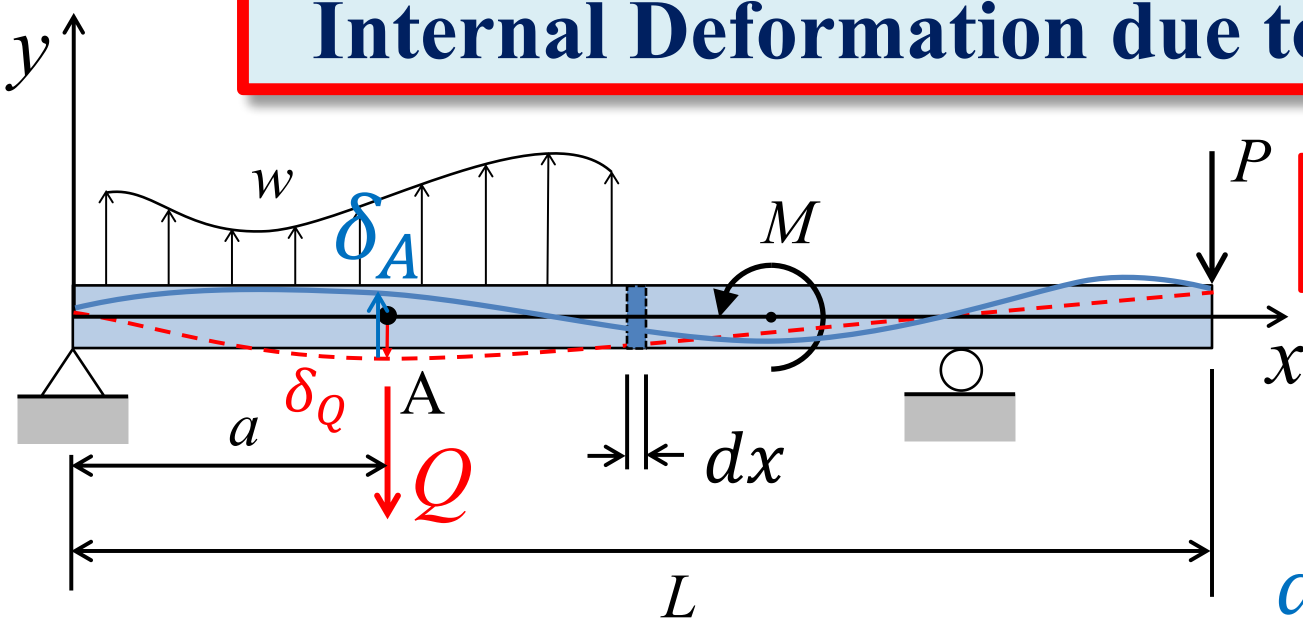
Modulus of Elasticity = $E$
Moment of Inertia = $I$




For bending, the internal virtual load is $M_Q$

# Add the Real Loads and Examine Internal Deformation due to Bending



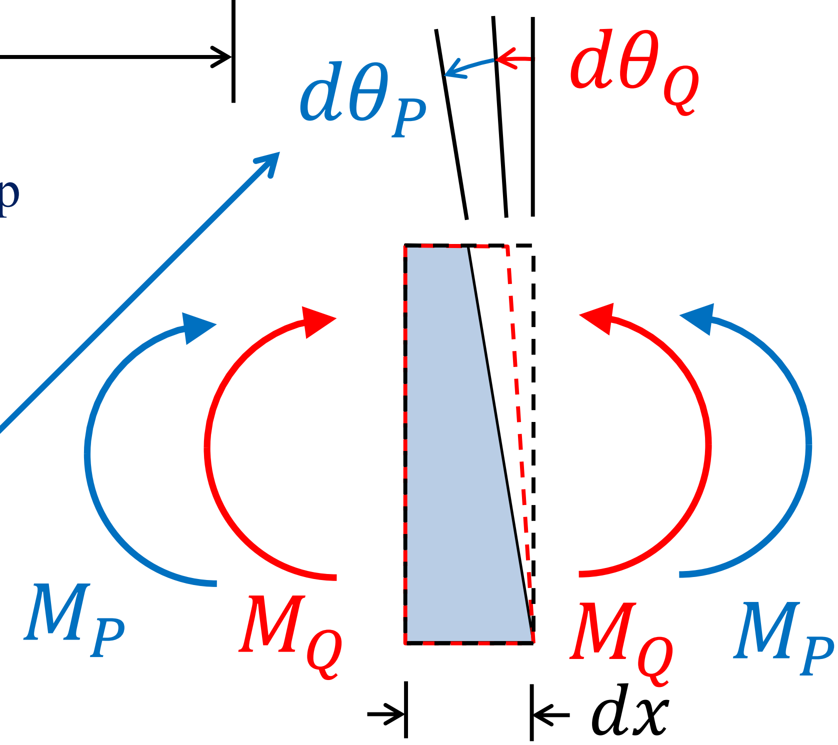

Modulus of Elasticity = $E$
Moment of Inertia = $I$

Can rewrite the moment-curvature relationship

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M_P}{EI}$$

$$\frac{d\theta_P}{dx} = \frac{M_P}{EI}$$

$$d\theta_P = \frac{M_P}{EI}dx$$



# Modify the General Form of the Principle of Virtual Work for Bending Deformation

Real Deformation

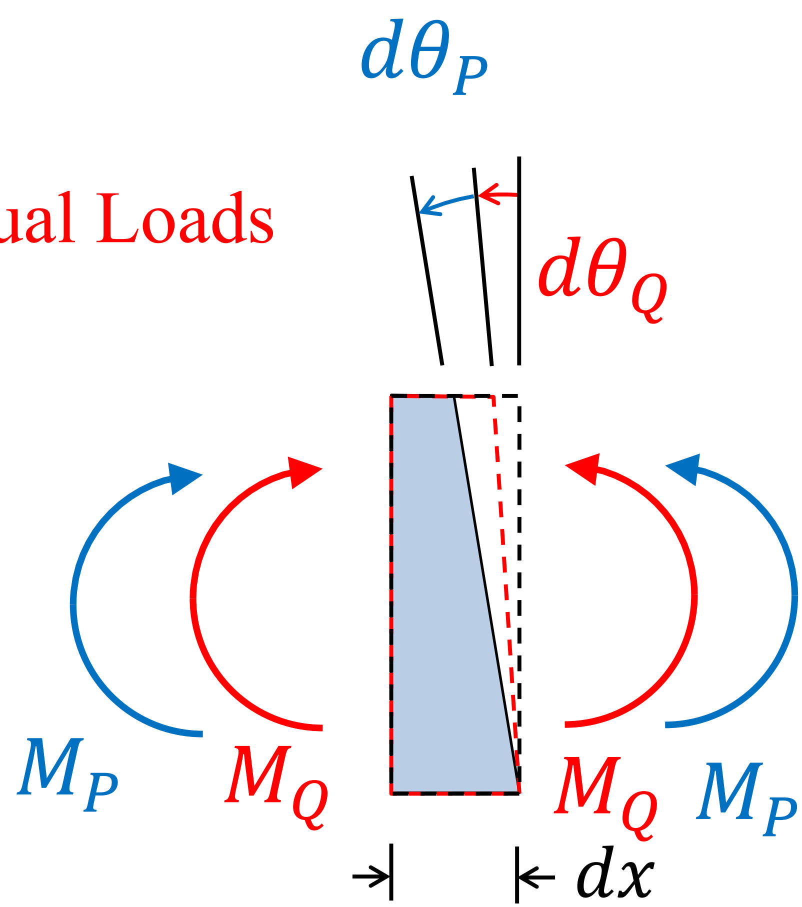
$$Q\delta_P = \iint F_Q dL_P$$

Virtual Loads

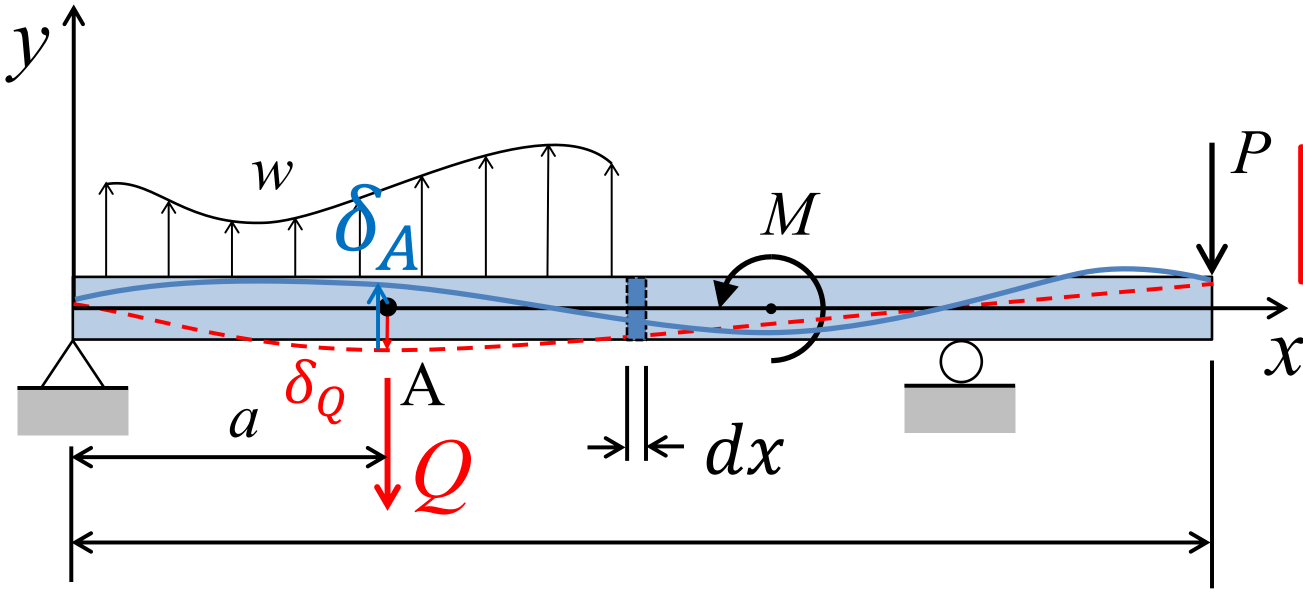
For bending deformation,
$dL_P = d\theta_P$ and $F_Q = M_Q$

The Principle of Virtual Work expression for bending can be expressed as:

$$Q\delta_P = \int_0^L M_Q d\theta_P$$



# Principle of Virtual Work to Measure $\delta_A$




Modulus of Elasticity = $E$
Moment of Inertia = $I$

$$Q\delta_P = \int_0^L M_Q d\theta_P$$

$$d\theta_P = \frac{M_P}{EI} dx$$

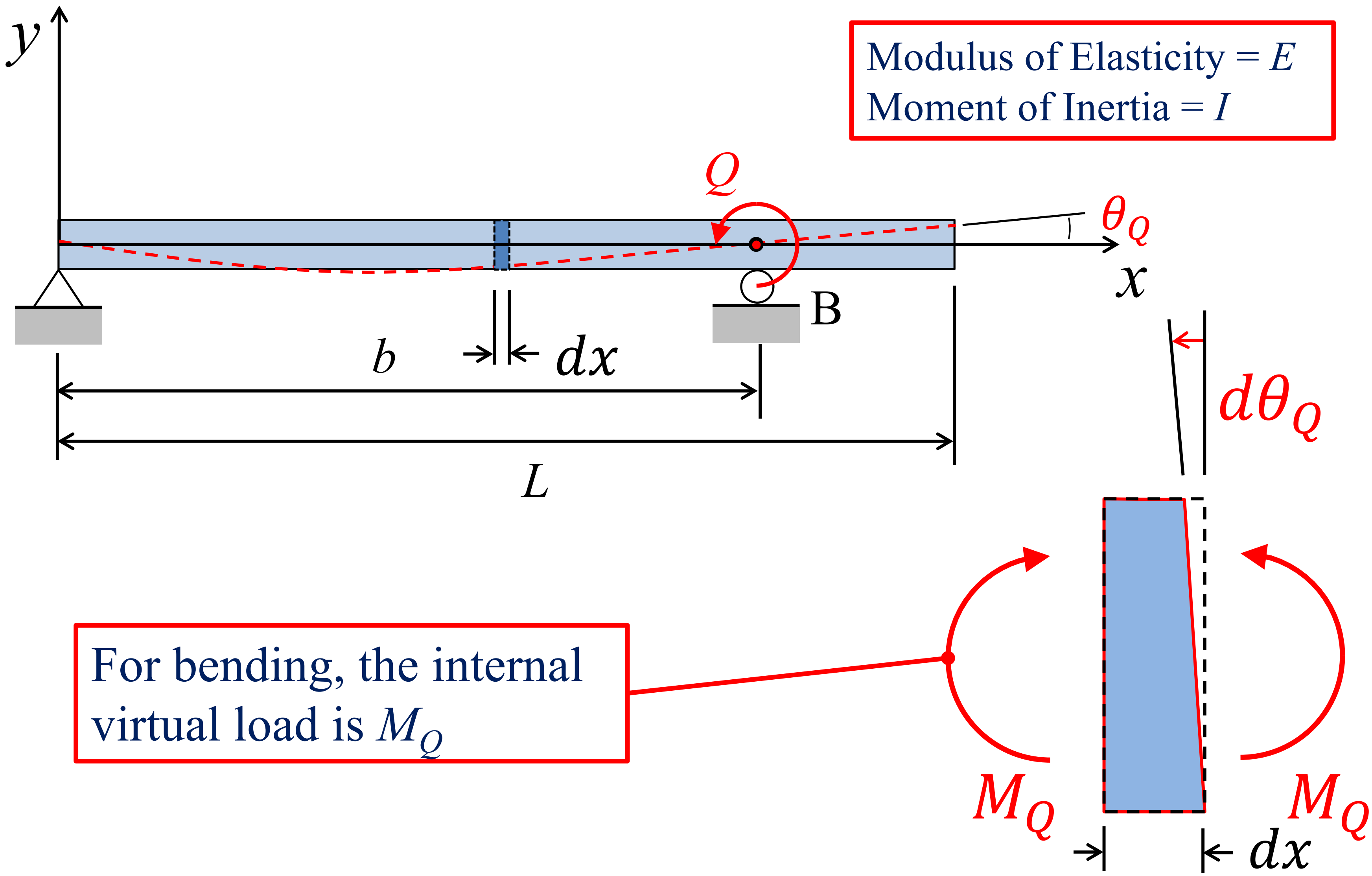
$$Q\delta_A = \int_0^L M_Q \frac{M_P}{EI} dx$$

If the bending stiffness, $EI$, is constant:

$$Q\delta_A = \frac{1}{EI} \int_0^L M_Q M_P dx$$

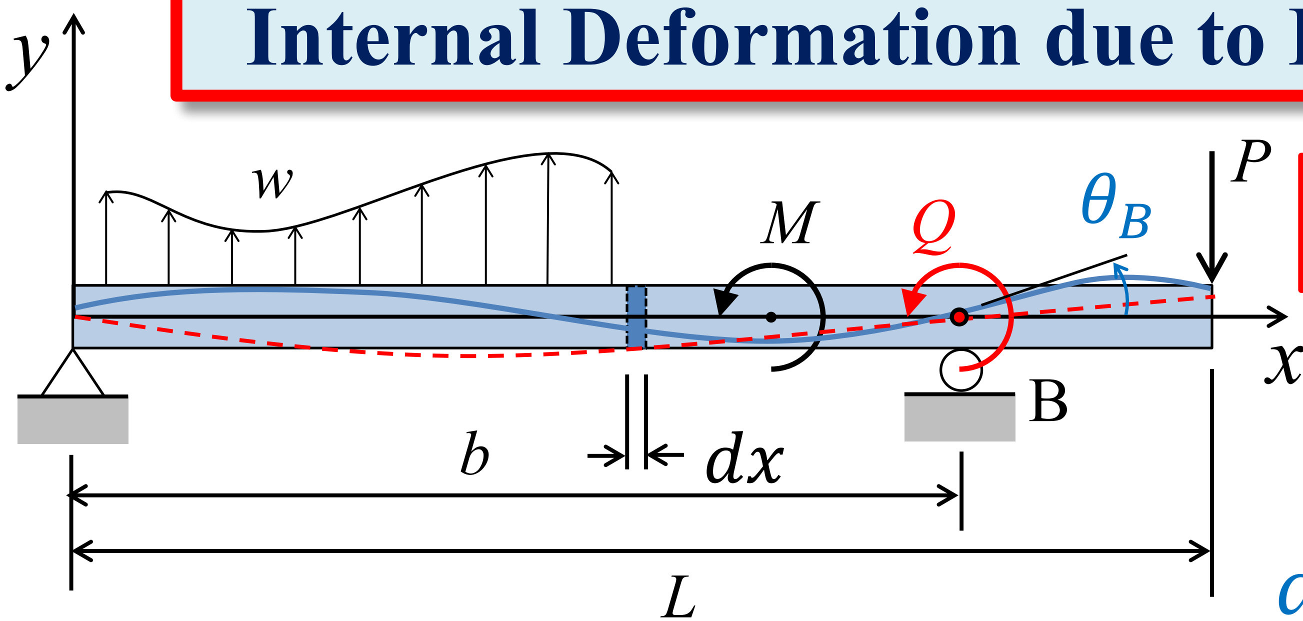
# To Measure Rotational Deformation, Apply a Virtual Moment

Modulus of Elasticity = $E$
Moment of Inertia = $I$




For bending, the internal virtual load is $M_Q$

# Add the Real Loads and Examine Internal Deformation due to Bending



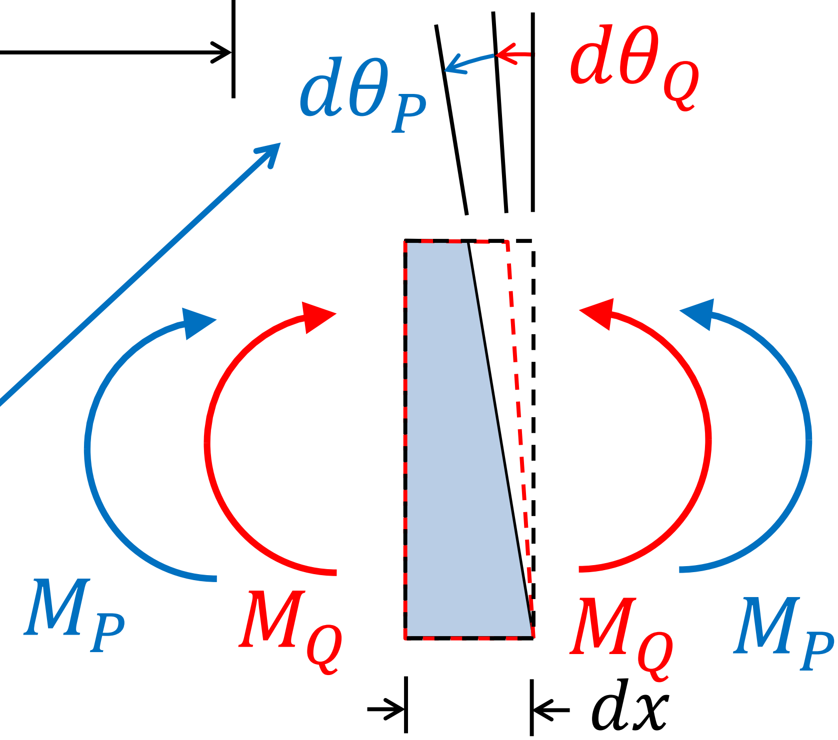

Modulus of Elasticity = $E$
Moment of Inertia = $I$

From the moment-curvature relationship

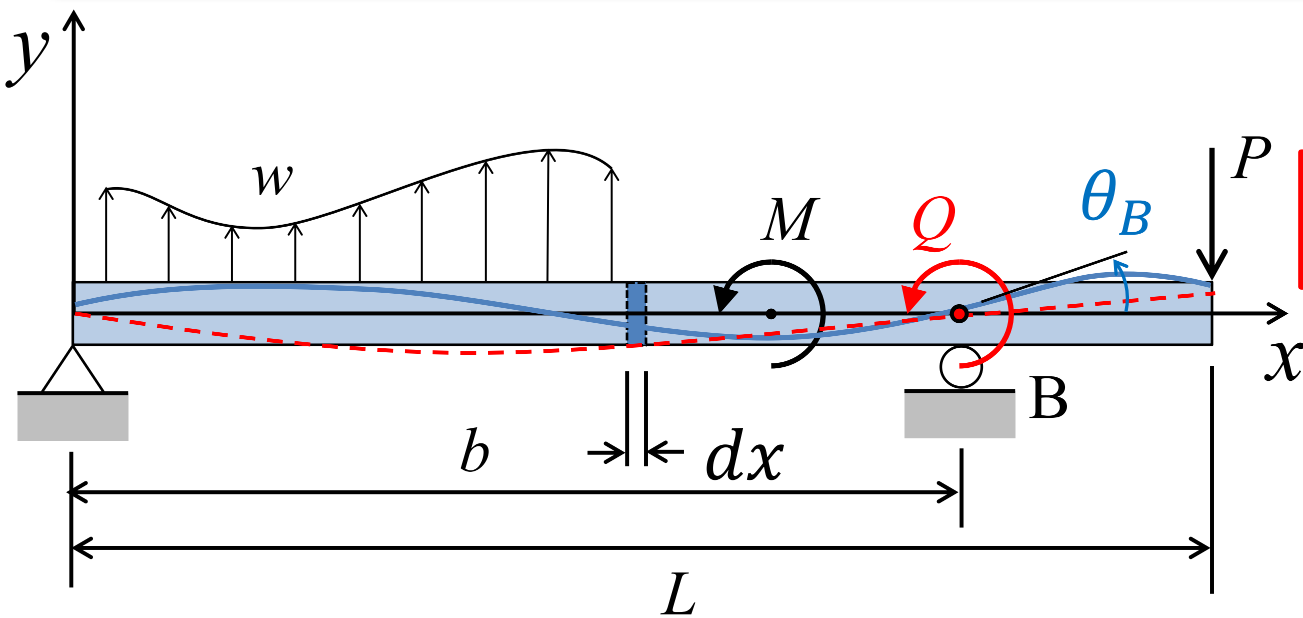
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M_P}{EI}$$

$$\frac{d\theta_P}{dx} = \frac{M_P}{EI}$$

$$d\theta_P = \frac{M_P}{EI}dx$$



# Principle of Virtual Work to Measure $\theta_A$




Modulus of Elasticity = $E$
Moment of Inertia = $I$

$$Q\delta_P = \int_0^L M_Q d\theta_P$$

$$d\theta_P = \frac{M_P}{EI} dx$$

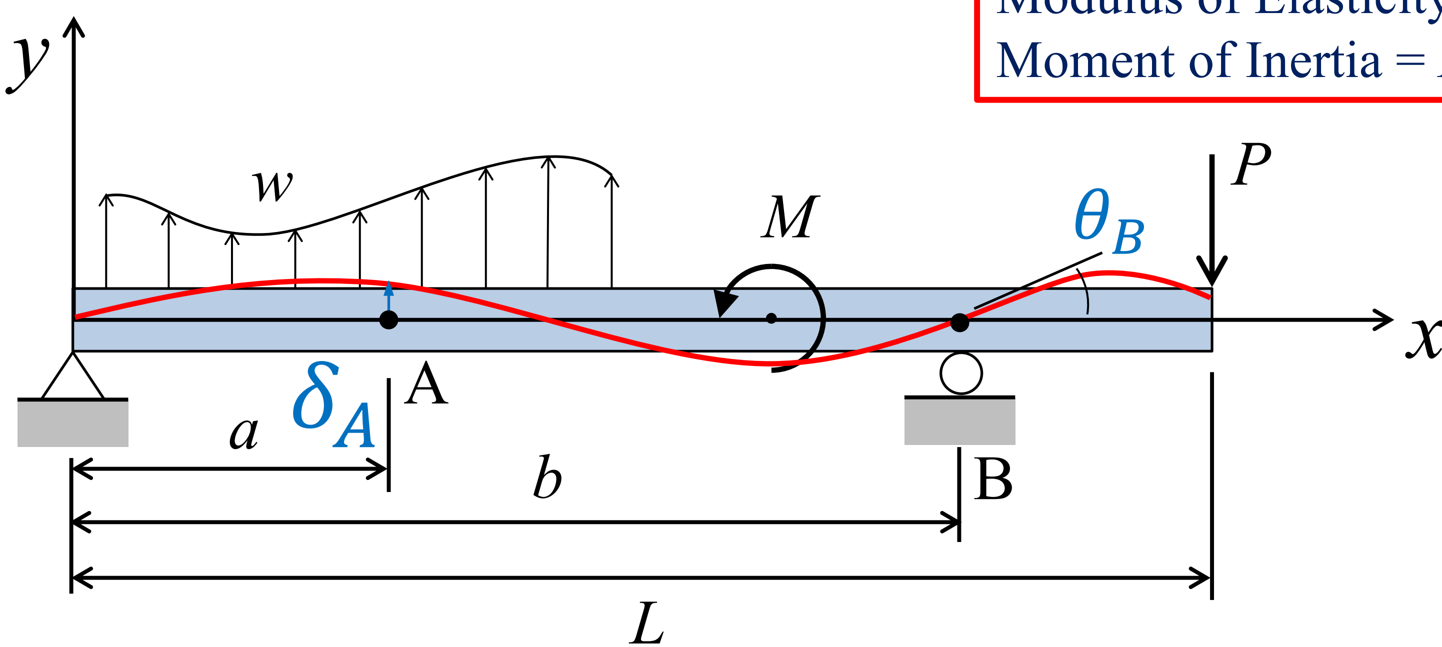
$$Q\theta_B = \int_0^L M_Q \frac{M_P}{EI} dx$$

If the bending stiffness, *EI*, is constant:

$$Q\theta_B = \frac{1}{EI}\int_0^L M_Q\, M_P dx$$

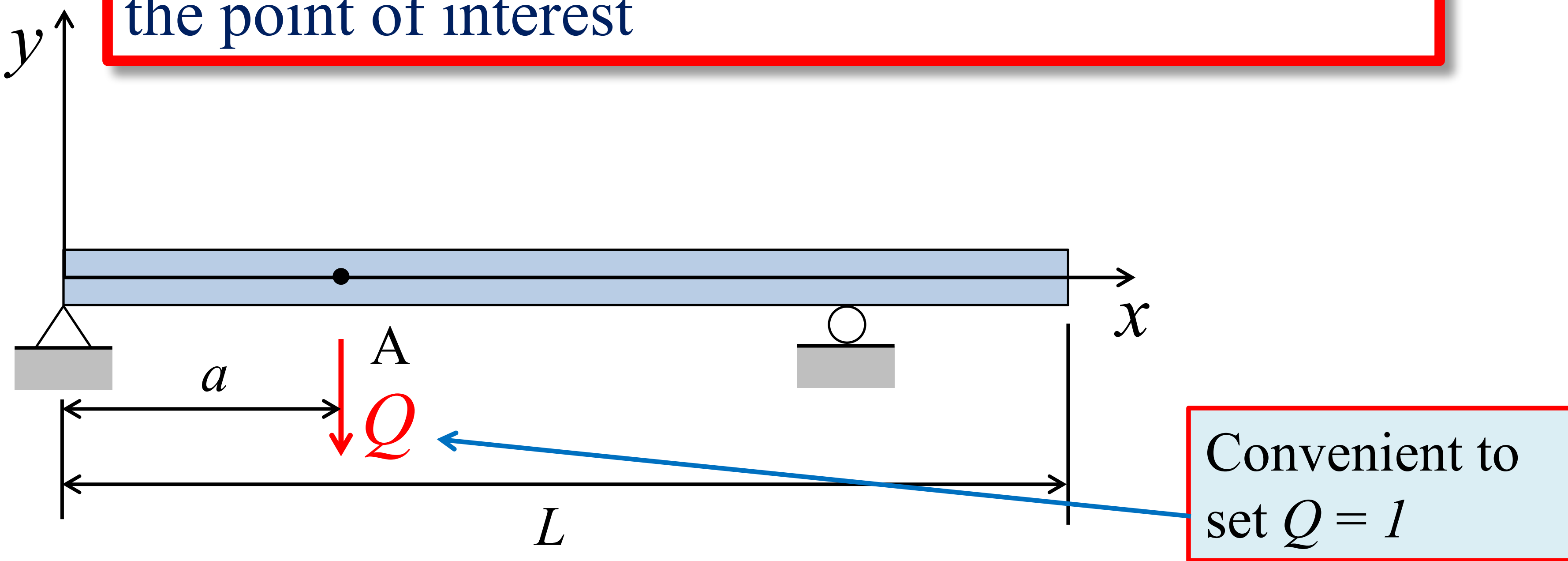
# Summary of Procedure for Finding Bending Deformation Using Virtual Work

Modulus of Elasticity = $E$
Moment of Inertia = $I$



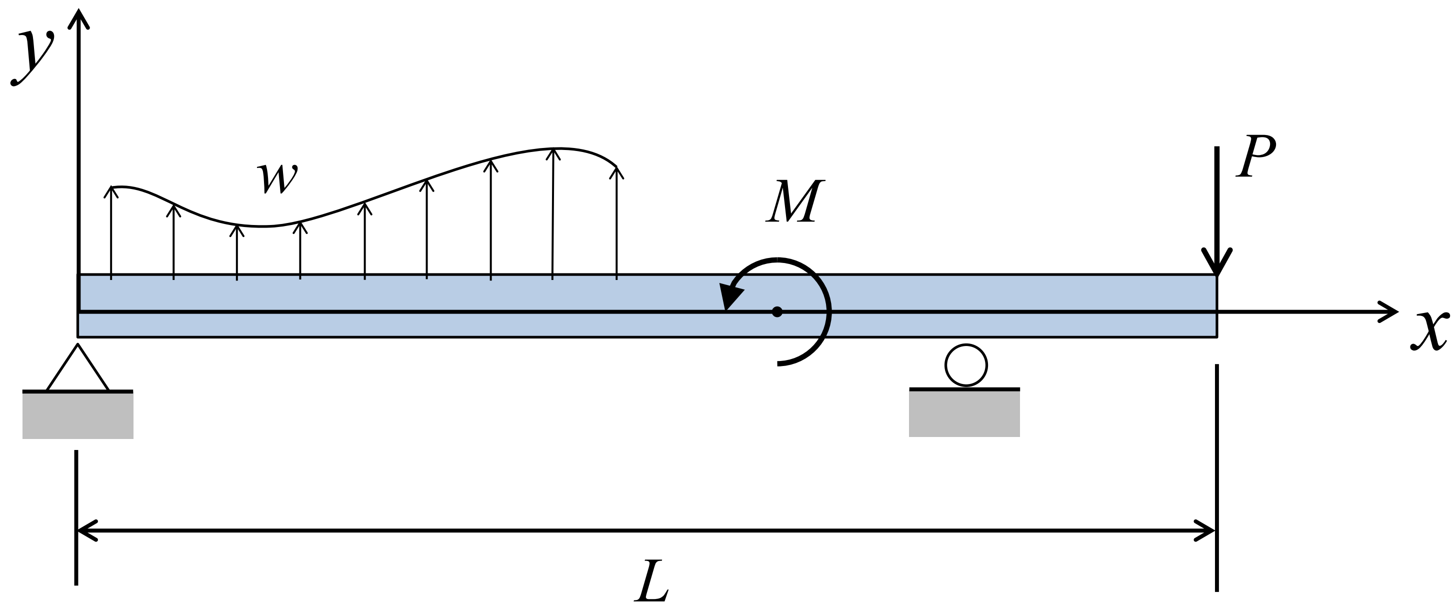

We want to find the deflection at point A and the slope at point B due to the applied loads

**Step 1** – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest




From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

**Step 2** – Replace all of the loads on the structure and perform the real analysis




From an equilibrium analysis, find the internal bending moment function for the real system: $M_P(x)$

## **Step 3** – Evaluate the virtual work product integrals and solve for the deformation of interest

$$Q\delta_A = \int_0^L M_Q \frac{M_P}{EI} dx$$

**If the bending stiffness, *EI*, is constant:**

$$Q\delta_A = \frac{1}{EI} \int_0^L M_Q \, M_P \, dx$$

**Table in textbook appendix is provided to help evaluate product integrals of this type**

# Table to Evaluate Virtual Work Product Integrals

**Appendix Table.2**

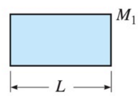
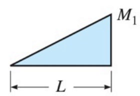
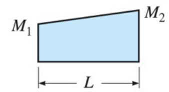
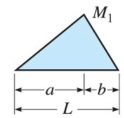
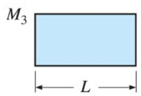
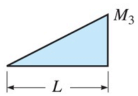
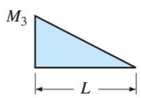
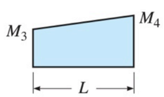
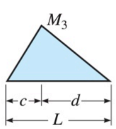
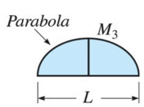
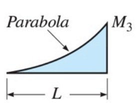
| $M_Q$ / $M_P$ |   |   |   |   |
|---|---|---|---|---|
|   | $M_1M_3L$ | $\frac{1}{2}M_1M_3L$ | $\frac{1}{2}(M_1+M_2)M_3L$ | $\frac{1}{2}M_1M_3L$ |
|   | $\frac{1}{2}M_1M_3L$ | $\frac{1}{3}M_1M_3L$ | $\frac{1}{6}(M_1+2M_2)M_3L$ | $\frac{1}{6}M_1M_3(L+a)$ |
|   | $\frac{1}{2}M_1M_3L$ | $\frac{1}{6}M_1M_3L$ | $\frac{1}{6}(2M_1+M_2)M_3L$ | $\frac{1}{6}M_1M_3(L+b)$ |
|   | $\frac{1}{2}M_1(M_3+M_4)L$ | $\frac{1}{6}M_1(M_3+2M_4)L$ | $\frac{1}{6}M_1(2M_3+M_4)L$ $+\frac{1}{6}M_2(M_3+2M_4)L$ | $\frac{1}{6}M_1M_3(L+b)$ $+\frac{1}{6}M_1M_4(L+a)$ |
|   | $\frac{1}{2}M_1M_3L$ | $\frac{1}{6}M_1M_3(L+c)$ | $\frac{1}{6}M_1M_3(L+d)$ $+\frac{1}{6}M_2M_3(L+c)$ | for $c \le a$: $\left(\frac{1}{3}-\frac{(a-c)^2}{6ad}\right)M_1M_3L$ |
|   | $\frac{2}{3}M_1M_3L$ | $\frac{1}{3}M_1M_3L$ | $\frac{1}{3}(M_1+M_2)M_3L$ | $\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$ |
|   | $\frac{1}{3}M_1M_3L$ | $\frac{1}{4}M_1M_3L$ | $\frac{1}{12}(M_1+3M_2)M_3L$ | $\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$ |

**Table is as useful tool to evaluate product integrals of the form:**

$$\int_0^L M_Q\, M_P\, dx$$