Beam Analysis by the Direct Stiffness Method

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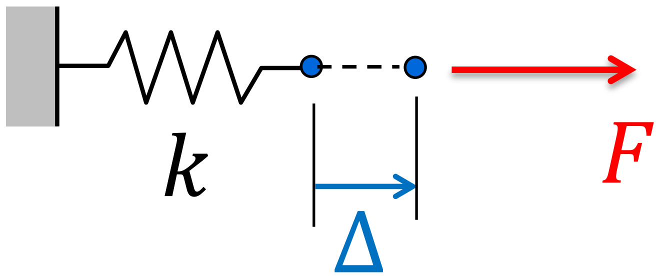
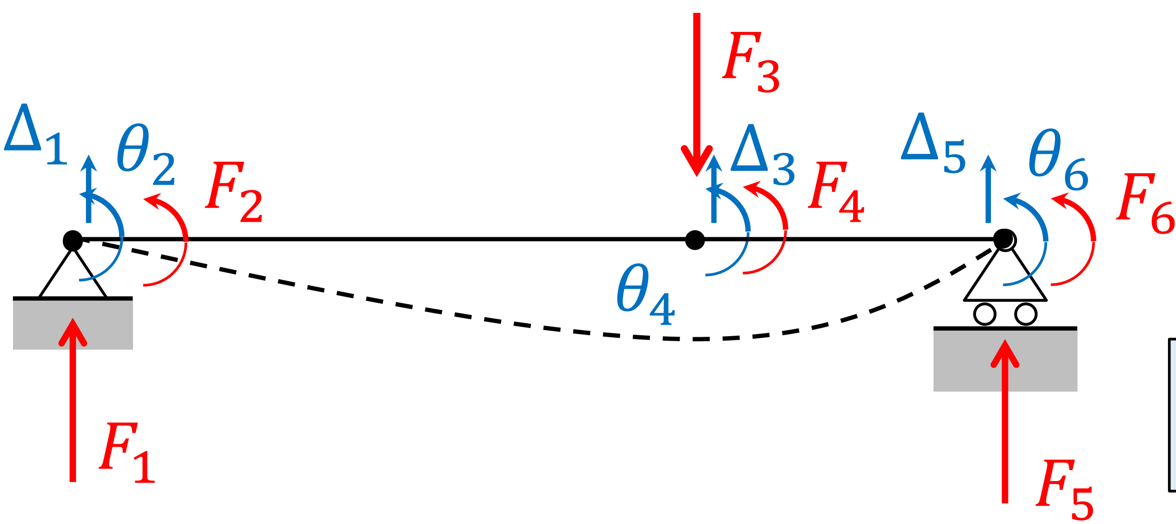
Stiffness Based Approach

Can think of a structure acting as a multi-dimensional spring



$$F = k\Delta$$



$$\{F\} = [K]\{\Delta\}$$

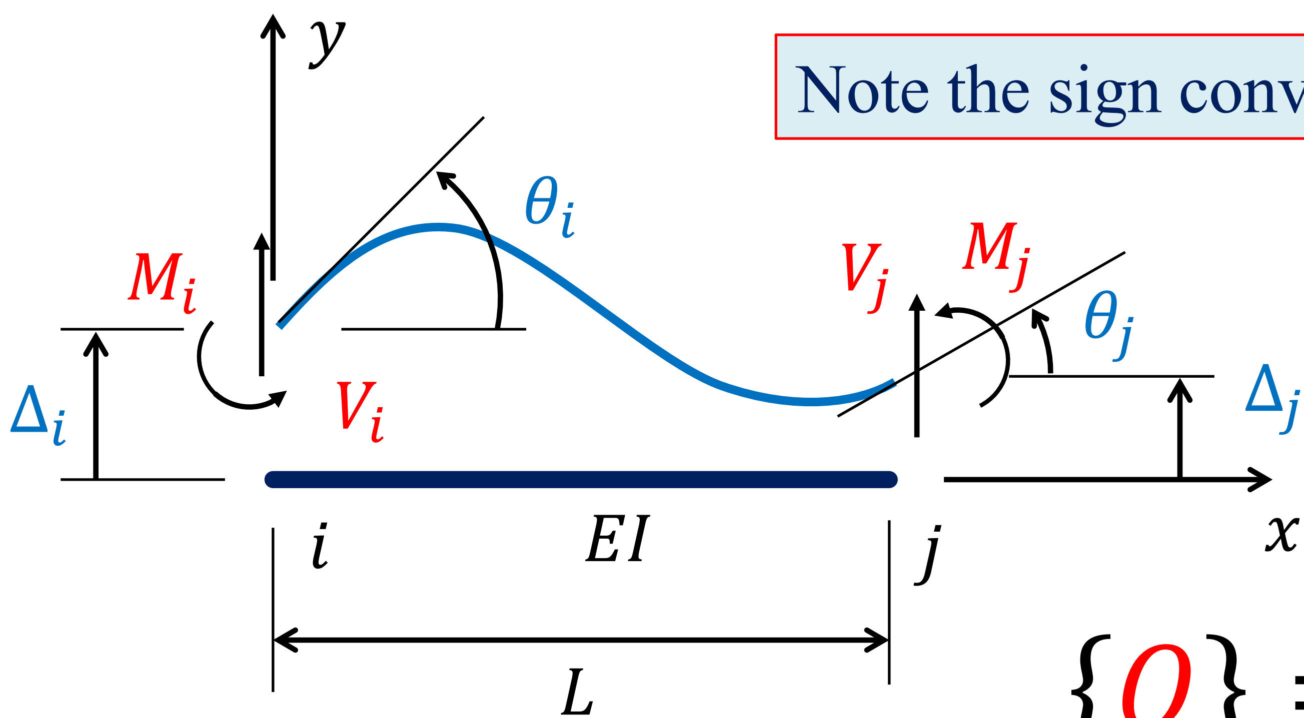
Beam Element Stiffness Matrix in Local Coordinates

Consider an inclined beam member with a moment of inertia I and modulus of elasticity E subjected to shear force and bending moment at its ends.

i = initial end of element
j = terminal end element

x axis (local 1 axis in SAP 2000)

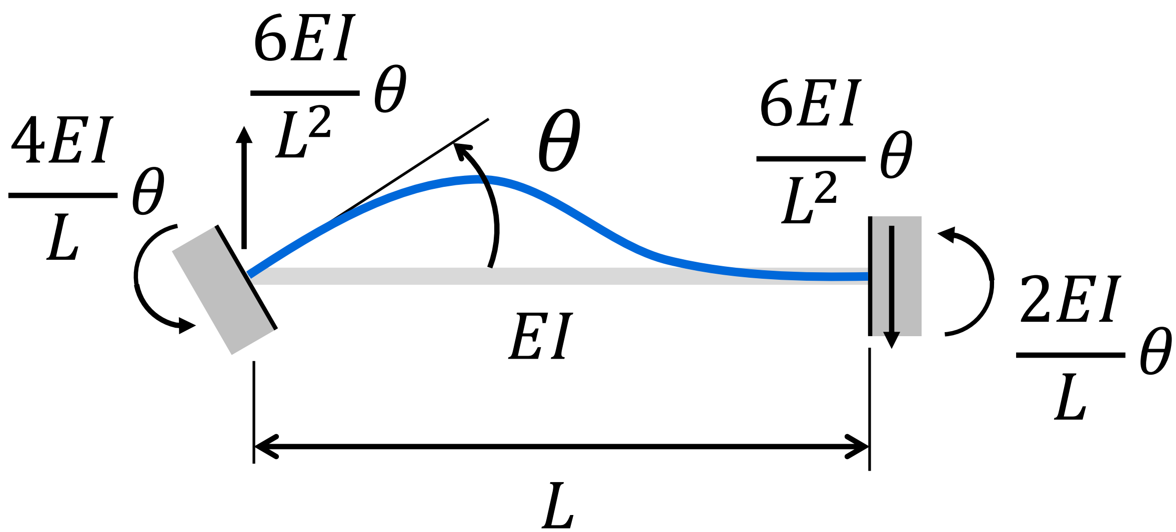
Note the sign convention

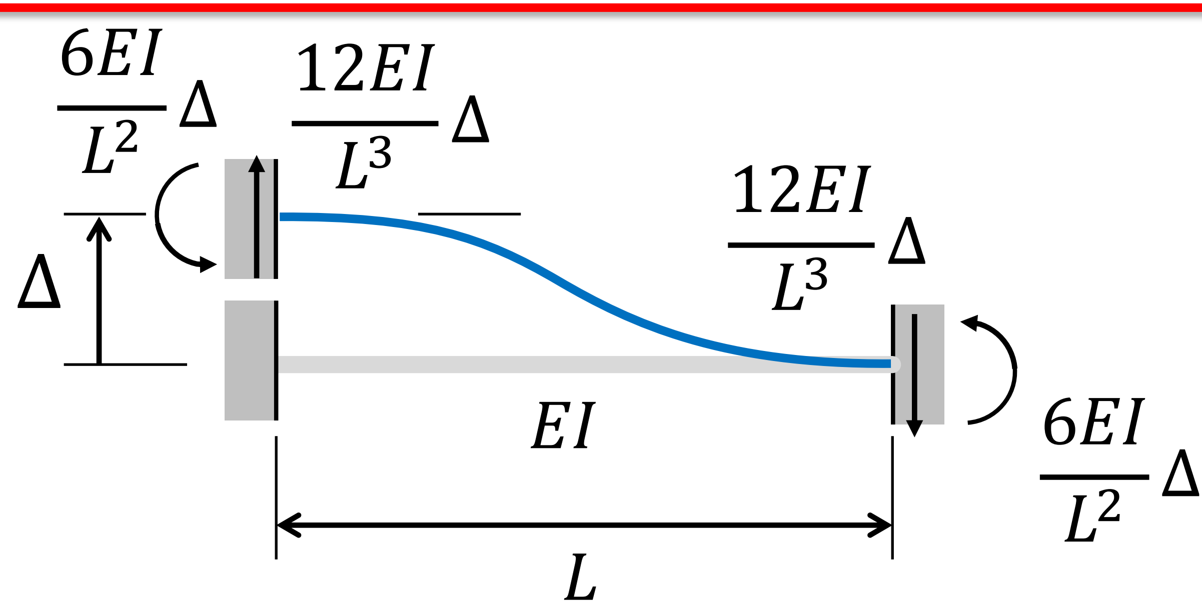


We want to find this 4x4 matrix

$$\{Q\} = [k]\{\delta\}$$

Each Column of the Frame Element Matrix in Local Coordinates is derived from Indeterminate Fixed End Moment Solutions

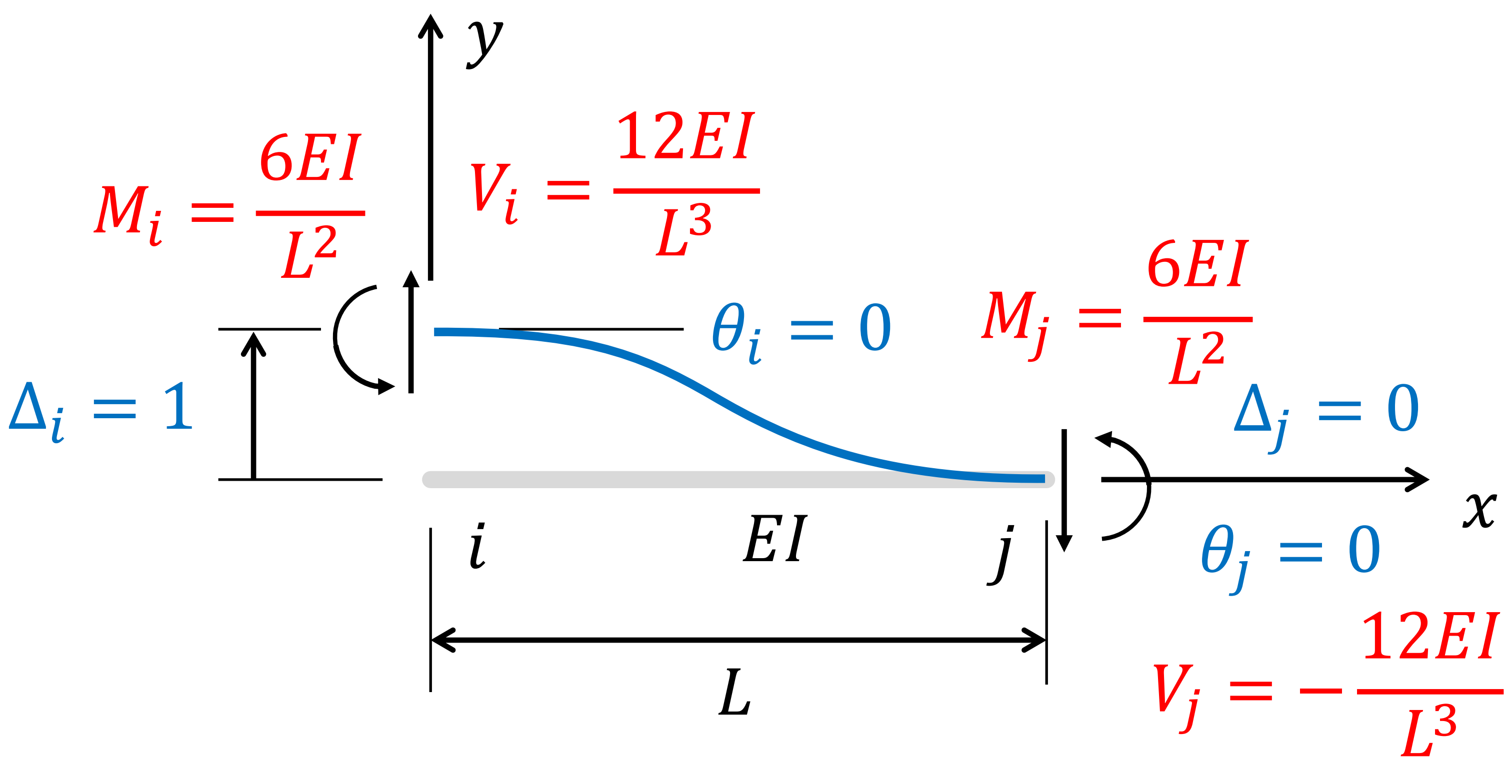




Find the First Column of the Frame Element Stiffness Matrix in Local Coordinates

set $\Delta_i = 1$

and $\theta_i = \theta_i = \Delta_j = 0$



Frame Element Stiffness Matrix

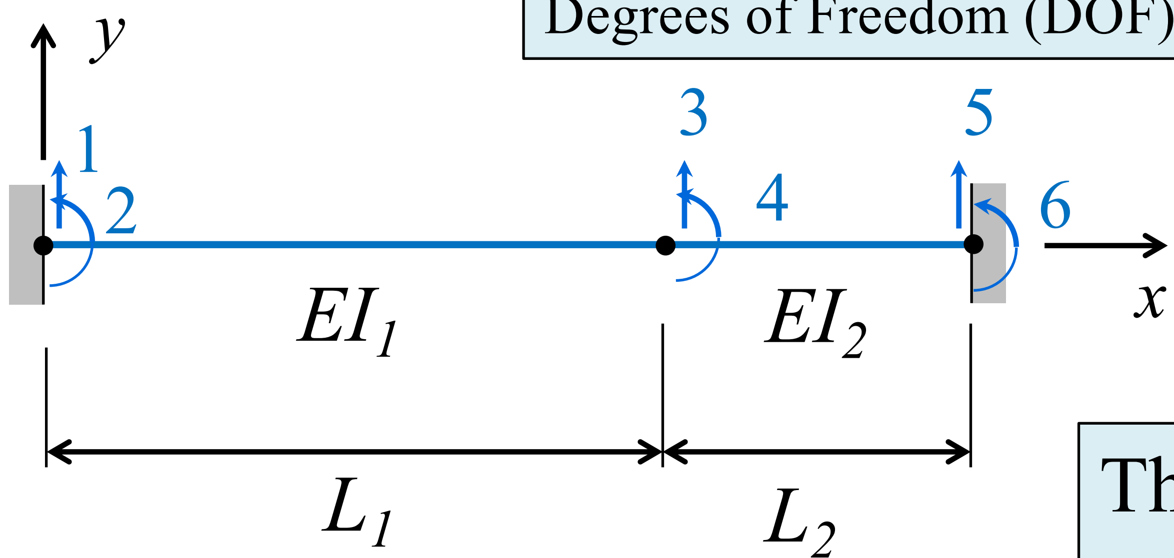
$$\begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \Delta_i \\ \theta_i \\ \Delta_j \\ \theta_j \end{Bmatrix}$$

$$\{Q\} = [k]\{\delta\}$$

Structure Stiffness Matrix

Consider a beam comprised of two elements

Each beam joint can move in two directions: 2 Degrees of Freedom (DOF) per joint

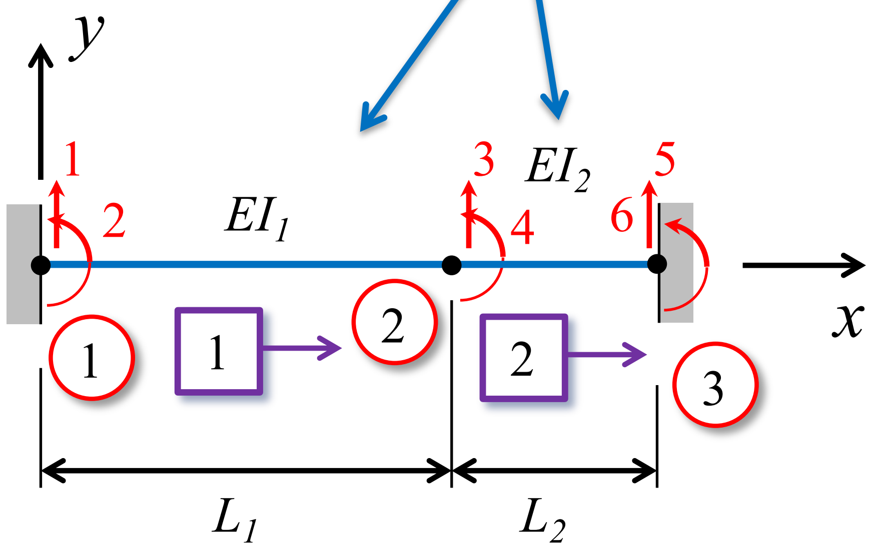


$$\{F\} = [K]\{\Delta\}$$

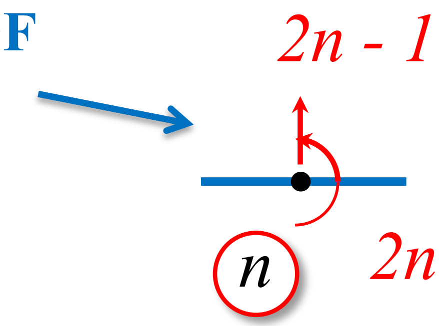
The 6x6 structure stiffness matrix can be assembled from the element stiffness matrices

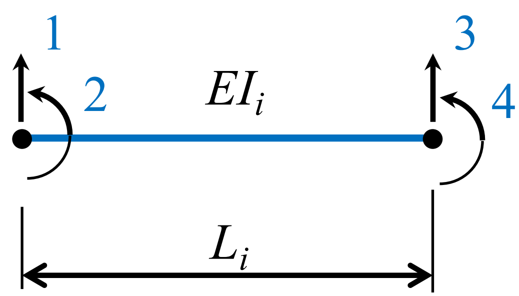
Assemble Structure Stiffness Matrix

1. Number each element



2. Joint and DOF numbering





3. Connectivity table to assemble structure stiffness matrix

	Element DOF			
	1	**2**	**3**	**4**
Associated global DOF for element 1	**1**	**2**	**3**	**4**
Associated global DOF for element 2	**3**	**4**	**5**	**6**

Assembly of Structure Stiffness Matrix from Element Contributions

Form 4x4 element stiffness matrix for element 1 from EI_1 and L_1

$[k]^1 =$

	1	2	3	4
1	k^1_{11}	k^1_{12}	k^1_{13}	k^1_{14}
2	k^1_{21}	k^1_{22}	k^1_{23}	k^1_{24}
3	k^1_{31}	k^1_{32}	k^1_{33}	k^1_{34}
4	k^1_{41}	k^1_{42}	k^1_{43}	k^1_{44}

Form 4x4 element stiffness matrix for element 2 from EI_2 and L_2

$[k]^2 =$

	3	4	5	6
3	k^2_{11}	k^2_{12}	k^2_{13}	k^2_{14}
4	k^2_{21}	k^2_{22}	k^2_{23}	k^2_{24}
5	k^2_{31}	k^2_{32}	k^2_{33}	k^2_{34}
6	k^2_{41}	k^2_{42}	k^2_{43}	k^2_{44}

Label the rows and columns of each 4x4 element stiffness matrix with corresponding structure DOF from connectivity table

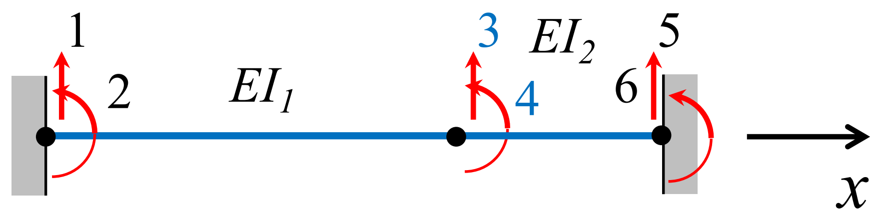
Assemble 6x6 structure stiffness matrix

$[K] =$

	1	2	3	4	5	6
1	k^1_{11}	k^1_{12}	k^1_{13}	k^1_{14}	0	0
2	k^1_{21}	k^1_{22}	k^1_{23}	k^1_{24}	0	0
3	k^1_{31}	k^1_{32}	$k^1_{33}+k^2_{11}$	$k^1_{34}+k^2_{12}$	k^2_{13}	k^2_{14}
4	k^1_{41}	k^1_{42}	$k^1_{43}+k^2_{21}$	$k^1_{44}+k^2_{22}$	k^2_{23}	k^2_{24}
5	0	0	k^2_{31}	k^2_{32}	k^2_{33}	k^2_{34}
6	0	0	k^2_{41}	k^2_{42}	k^2_{43}	k^2_{44}

Structure System of Equations: Free DOF

DOF 3 are 4 are free DOF;



DOF 1, 2, 5,and 6 are restrained (support) DOF

$$\begin{Bmatrix} V_1 \\ M_2 \\ V_3 \\ M_4 \\ V_5 \\ M_6 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \theta_2 \\ \Delta_3 \\ \theta_4 \\ \Delta_5 \\ \theta_6 \end{Bmatrix}$$

At free DOF (blue), we know the forces (applied joint loads) but the displacements are unknown

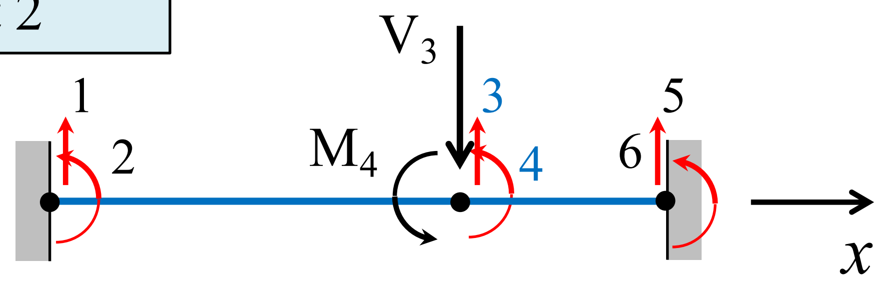
At restrained DOF, we know the displacements but the forces (support reactions) are unknown

Free DOF System of Equations

Suppose we have loads applied to joint 2

DOF 3 are 4 are free DOF;

DOF 1, 2, 5,and 6 are restrained (support) DOF

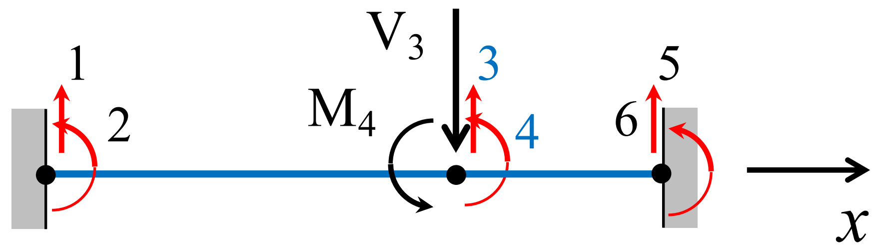


$$\begin{Bmatrix} V_1 \\ M_2 \\ V_3 \\ M_4 \\ V_5 \\ M_6 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Delta_3 \\ \theta_4 \\ 0 \\ 0 \end{Bmatrix}$$

At free DOF (blue), we know the forces (applied joint loads) but the displacements are unknown

At restrained DOF, we know the displacements (all equal to zero) but the forces (support reactions) are unknown

Solve for Displacements at Free DOF



Free DOF Equation Set

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} \Delta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} V_3 \\ M_4 \end{Bmatrix}$$

Solve for Δ_3 and θ_4

Find Support Reactions

DOF 1, 2, 5 and 6 are restrained (support) DOF

$$\begin{Bmatrix} V_1 \\ M_2 \\ V_3 \\ M_4 \\ V_5 \\ M_6 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Delta_3 \\ \theta_4 \\ 0 \\ 0 \end{Bmatrix}$$

Δ_3 and θ_4, found from previous step

After displacements are found, multiply to find unknown forces (support reactions)

$$\begin{Bmatrix} V_1 \\ M_2 \\ V_5 \\ M_6 \end{Bmatrix} = \begin{bmatrix} K_{13} & K_{14} \\ K_{23} & K_{24} \\ K_{53} & K_{54} \\ K_{63} & K_{64} \end{bmatrix} \begin{Bmatrix} \Delta_3 \\ \theta_4 \end{Bmatrix}$$