METR 130: Lecture 4

- Reynolds Averaged Conservation Equations
- Turbulent Fluxes (Definition and typical ABL profiles, CBL and SBL)
- Turbulence Closure Problem & Parameterization

Spring Semester 2011

April 5, 2011

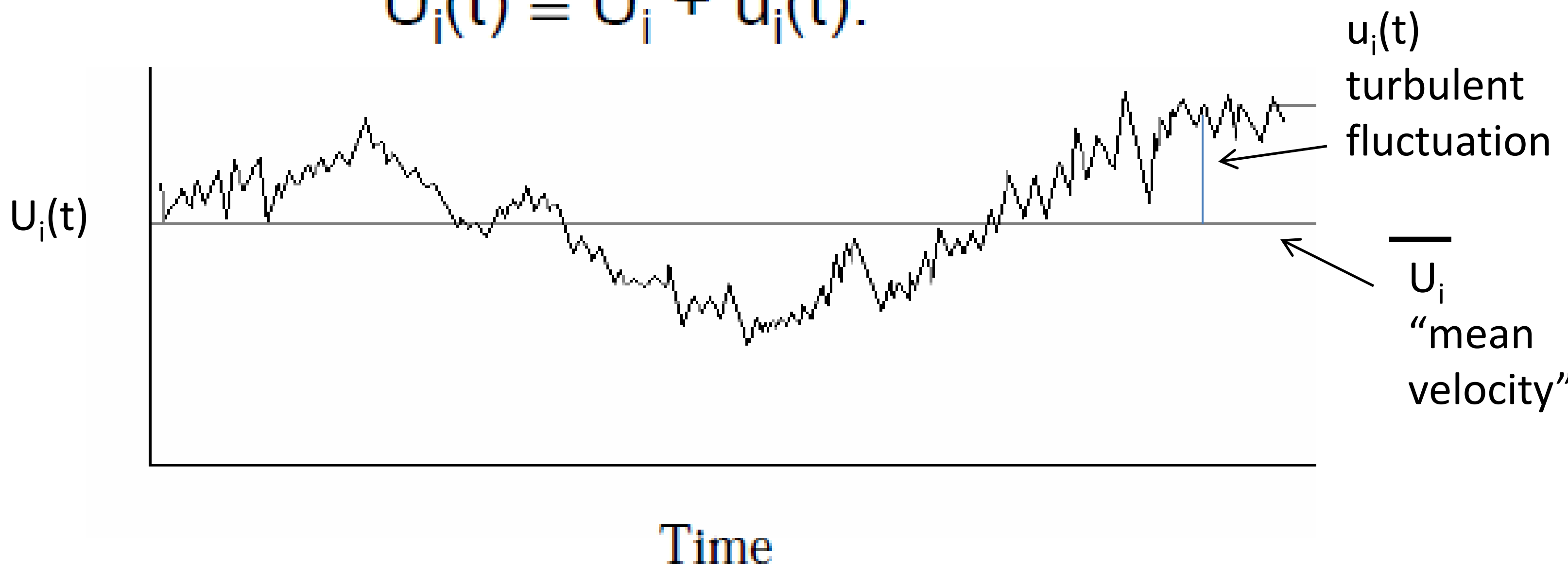
Reading from Arya

- **Chapters 5.4 & 5.5**
- **Chapter 6**
  - **6.1 through 6.3 (Review)**
  - **6.4 & 6.5 (note Richardson number vs. height in stable BL)**
- **Chapter 8.1 through 8.5**
- **Chapter 9.1 & 9.2**
- Chapter 13
  - Not required, but maybe helpful.
  - Some advanced topics related to parameterization.
  - Page 287 ("integral models") has some material relevant to Assignment #3 Problem 3.

Turbulence Decomposition of Velocity

(See also 8.4 of Arya) …

$$U_i(t) \equiv \overline{U_i} + u_i(t).$$



Similar decomposition for other variables …

1) Potential Temperature
2) Specific Humidity
3) Species Concentration
4) Pressure
5) Density (although, can relate to P & T through IGL)

Reynolds Averaging Postulates
(or results based on these …)

Let ‘A’ and ‘B’ be variables, and ‘c’ be a constant

$$\overline{\overline{A}} = \overline{A}$$

$$\overline{A'} = 0$$

$$\overline{cA} = c\overline{A}$$

$$\overline{A+B} = \overline{A} + \overline{B}$$

$$\overline{AB} \neq \overline{A}\,\overline{B}$$

$$\overline{\overline{A}\,\overline{B}} = \overline{A}\,\overline{B}$$

$$\overline{\frac{\partial A}{\partial x}} = \frac{\partial \overline{A}}{\partial x}$$

Space for any derivations, math to show that these are true …

Starting Point …
(u-momentum equation)

- Combine U momentum equation and incompressible form of continuity equations …

$$\frac{\partial u}{\partial t} = -\frac{\partial(uu)}{\partial x} - \frac{\partial(vu)}{\partial y} - \frac{\partial(wu)}{\partial z} - \frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)$$

Advection (written in "flux" form)

Pressure Gradient Force

coriolis force

Viscosity

- $\rho \approx \rho_0$ = constant via "Boussenesq" assumption
- Equation above is for <u>instantaneous</u> flow.

Ending Point …

(Reynolds Averaged u-momentum equation)

- Decompose variables as $\overline{(\quad)} + (\quad)'$
- Reynolds Average both sides of equation …

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial(\overline{u}\,\overline{u})}{\partial x} - \frac{\partial(\overline{v}\,\overline{u})}{\partial y} - \frac{\partial(\overline{w}\,\overline{u})}{\partial z} - \frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\overline{v} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right) - \frac{\partial(\overline{u'u'})}{\partial x} - \frac{\partial(\overline{u'v'})}{\partial y} - \frac{\partial(\overline{u'w'})}{\partial z}$$

Mean Advection (written in "flux" form)

Pressure Gradient Force (Mean)

Coriolis Force (mean)

Mean Viscosity

Divergence of turbulent u momentum flux. (NEW TERMS)

- Above equation is for the Reynolds-averaged (or mean) u velocity.
- $\rho \approx \rho_0$ = constant via "Boussenesq" assumption.
- Viscosity term can be shown to be small in most flows of geophysical interest (meteorological, oceanographic)
- Above equation w/out viscosity term is essentially the form of the u-momentum equation used in 3-D weather & climate models

Boundary Layer Form of Equation …

(i.e. after making "boundary-layer" assumption)

$$\frac{\partial}{\partial z} >> \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial y}$$

$$\frac{\partial \overline{u}}{\partial t} = f\left(\overline{v} - v_g\right) - \frac{\partial(\overline{u'w'})}{\partial z}$$

Pressure Gradient & Coriolis Forces

Divergence of vertical turbulent u momentum flux.

- Wrote PGF in above equation in terms of geostrophic wind
- BL Assumption alternatively can be viewed as an assumption of horizontal homogenieity
- Horizontal Homogeneity – statistics of variables do not vary horizontally.
- Horizontal homogeneity implies through incompressible continuity equation that w = 0.
- Above equation is the form of the u-momentum equation used for the basic boundary layer research and testing of parameterizations.

"Closure Problem" …

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial(\overline{u}\,\overline{u})}{\partial x} - \frac{\partial(\overline{v}\,\overline{u})}{\partial y} - \frac{\partial(\overline{w}\,\overline{u})}{\partial z} - \frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\overline{v} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right) \underbrace{- \frac{\partial(\overline{u'u'})}{\partial x} - \frac{\partial(\overline{u'v'})}{\partial y} - \frac{\partial(\overline{u'w'})}{\partial z}}$$

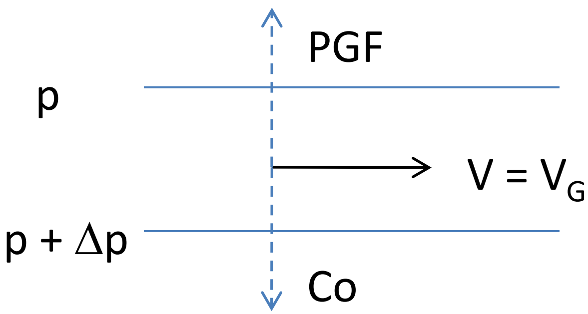
Divergence of turbulent u momentum flux. (NEW TERMS!)

- New terms involving turbulence fluxes introduce <u>additional unknowns</u>
- Similar terms get introduced when going through the procedure for other equations (e.g. v, θ, q)
- However since no new equations have been introduced into the system … system is <u>unclosed</u>
- An unclosed system of equations cannot be solved
- Need to represent unclosed terms in terms of known variables (i.e. those that we have equations for) in order to solve system
- i.e. … we require "turbulence parameterizations" for the turbulence fluxes.
- Will be seen how to do this later on …

Also … Remember from Lecture 1

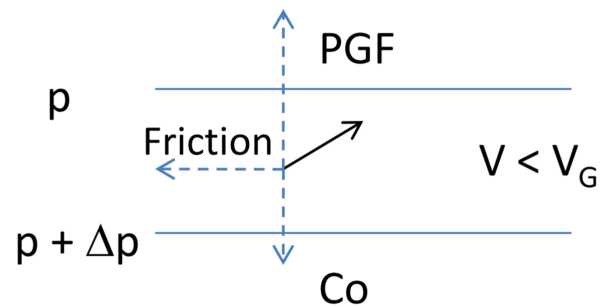
(Also see Arya, Chapter 6)

Above the boundary layer
(two main forces: PGF and CO)



Wind is geogrophic …
(or perhaps “gradient flow”
or something in between);
main point: no friction, wind
parallel to isobars

Near the surface
(three main forces: PGF, CO & Friction)



… Wind slowed due to friction.
Wind flow at angle α_0 to isobars
(“cross isobaric flow angle”)

Momentum Equations: ABL

$$f(v - v_g) + \frac{d}{dz}\left(\frac{\tau_x}{\rho}\right) = 0,$$

$$-f(u - u_g) + \frac{d}{dz}\left(\frac{\tau_y}{\rho}\right) = 0.$$

Equations for **mean** velocity
(Note three forces, which is "friction"?)

Divergence of vertical **turbulent shear stress** per unit mass, where τ_x = x-component of vertical turbulent shear stress and τ_y = y-component of vertical turbulent shear stress.

- These are the "F" terms used in MET121 for the friction force.
- Magnitude of shear stress = $(\tau_x^2 + \tau_y^2)^{1/2} \equiv \tau$
- Surface value $\tau(z=0)/\rho = \tau_0/\rho = u_*^2$
- The implied key velocity scale u_* is called the "**friction velocity**"
- **NEW UNDERSTANDING**: $\tau_x/\rho = -\overline{u'w'}$ and $\tau_y/\rho = -\overline{v'w'}$ (i.e. stress = flux)

Full Boundary Layer Equations …

$$\frac{\partial \overline{u}}{\partial t} = f\left(\overline{v} - v_g\right) - \frac{\partial\left(\overline{u'w'}\right)}{\partial z}$$

$$\frac{\partial \overline{v}}{\partial t} = -f\left(\overline{u} - u_g\right) - \frac{\partial\left(\overline{v'w'}\right)}{\partial z}$$

Divergence of vertical turbulent fluxes of u and v velocity.

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial\left(\overline{w'\theta'}\right)}{\partial z} + S_\theta^+ - S_\theta^-$$

$$\frac{\partial \overline{q}}{\partial t} = -\frac{\partial\left(\overline{w'q'}\right)}{\partial z} + S_q^+ - S_q^-$$

$$\frac{\partial \overline{\chi}}{\partial t} = -\frac{\partial\left(\overline{w'\chi'}\right)}{\partial z} + +S_\chi^+ - S_\chi^-$$

HOMEWORK: Derive one of these three equations.

Divergence of vertical turbulent fluxes of heat (θ), moisture (q) and a pollutant species (χ).

In above equations …

- *u_g and v_g are geostrophic wind speed components*
- *S^+ and S^- are source and sink terms, respectively*

Reynolds Stress Tensor

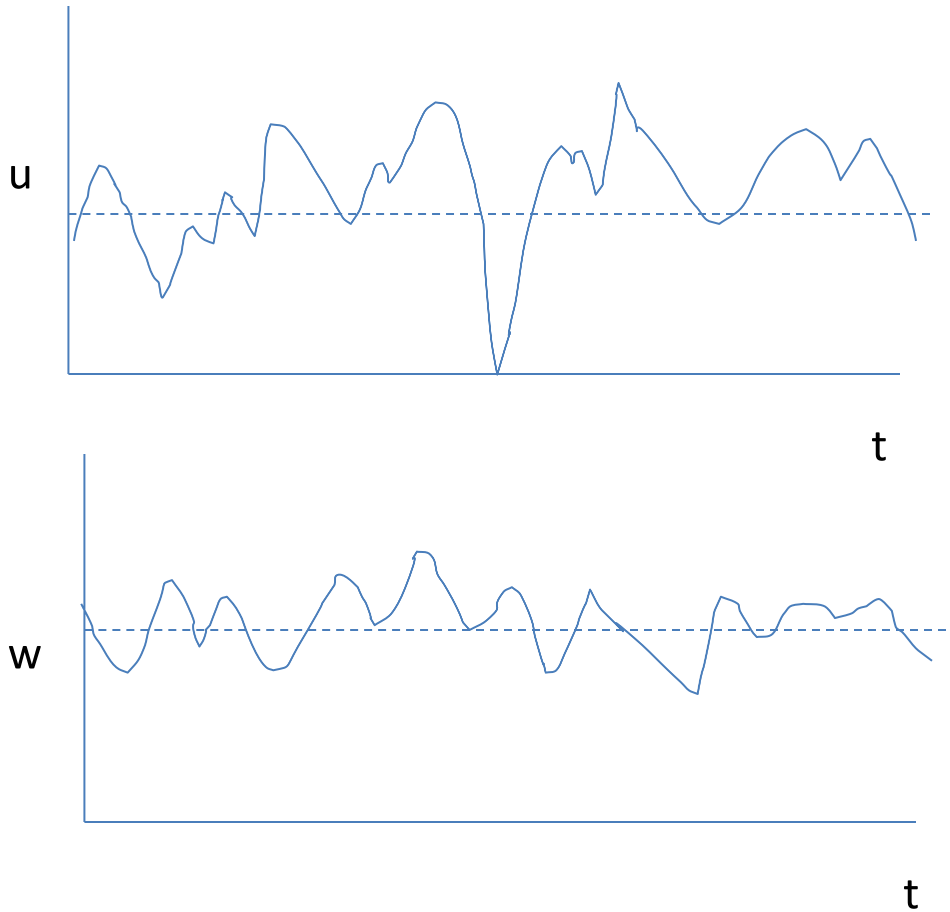
$$\overline{u_i' u_j'} = \begin{vmatrix} \overline{u_1' u_1'} & \overline{u_1' u_2'} & \overline{u_1' u_3'} \\ \overline{u_2' u_1'} & \overline{u_2' u_2'} & \overline{u_2' u_3'} \\ \overline{u_3' u_1'} & \overline{u_3' u_2'} & \overline{u_3' u_3'} \end{vmatrix} = \begin{vmatrix} \overline{u'^2} & \overline{u' v'} & \overline{u' w'} \\ \overline{u' v'} & \overline{v'^2} & \overline{v' w'} \\ \overline{u' w'} & \overline{v' w'} & \overline{w'^2} \end{vmatrix}$$

- i = 1, 2 and 3 & j = 1, 2 and 3 are components of fluctuating velocity vector u'_i and u'_j
- Far RHS: set $u'_1 = u'$, $u'_2 = v'$ and $u'_3 = w'$ (typical meteorological coordinates)
- Sum of diagonal components = $\overline{u'^2} + \overline{v'^2} + \overline{w'^2} = \sigma^2$ = turbulent velocity variance
- $\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)/2 = \sigma^2/2$ = “turbulent kinetic energy”

Turbulence Fluxes

(Random Example)

Again, use $\overline{u'w'}$ as an example …



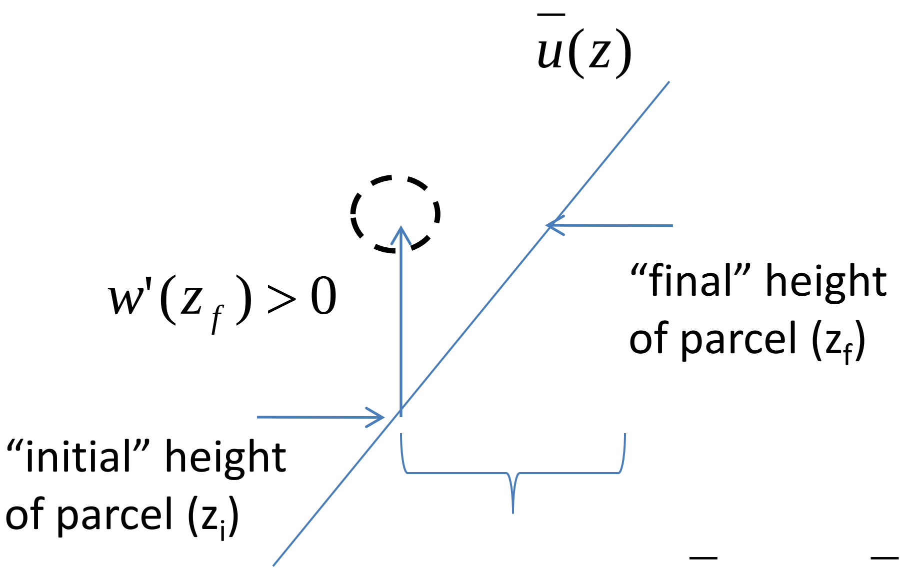
In this example,
I have drawn the u and
w traces completely
Randomly and completely
Independent of each
In this case, we say that
u’ and w’ are “uncorrelated”
And $\overline{u'w'} = 0$.

$$R_{uw} = \frac{\overline{u'w'}}{\sigma_u \sigma_w} = 0$$

“correlation coefficient”
for $\overline{u'w'}$. In this case, $R_{uw} = 0$.

Fluxes tend to be correlated in ABL due to non-uniform mean profiles

(e.g. mean wind shear)



$$u'(z_f) = u(z_f) - \overline{u}(z_f) = \overline{u}(z_i) - \overline{u}(z_f) < 0$$

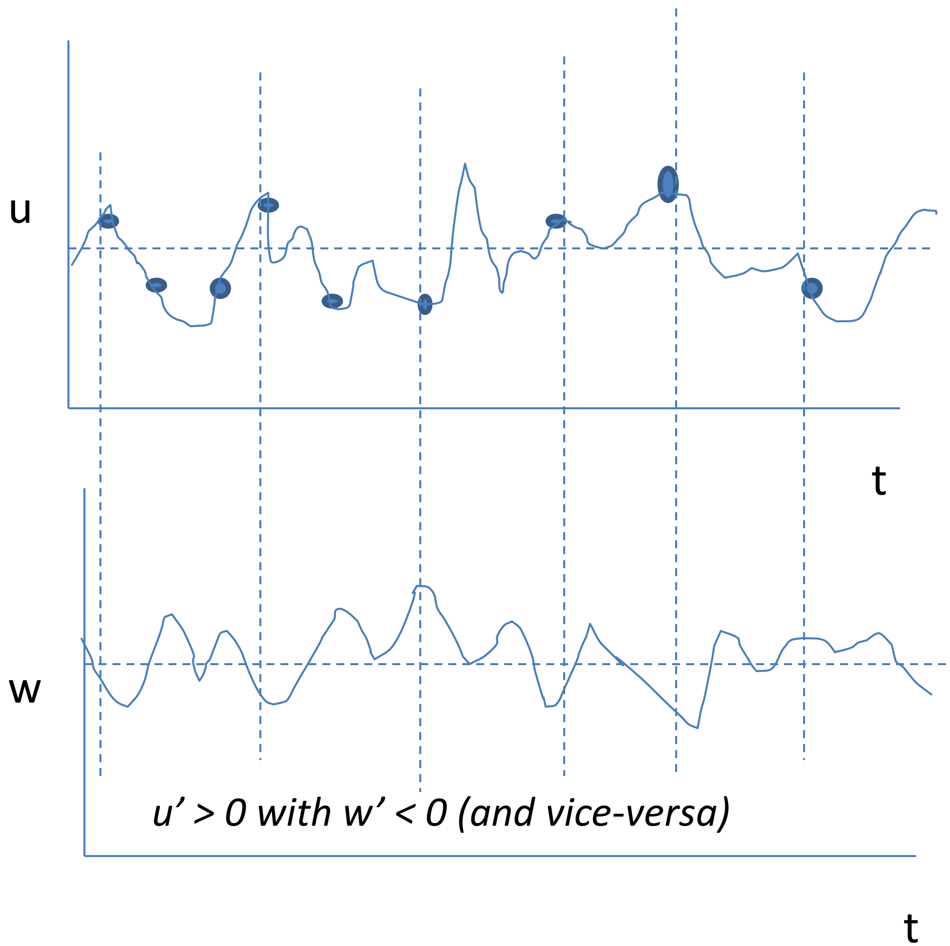
therefore w’ > 0 associated with u’ < 0 (negatively correlated).

Can be shown (diagram for yourself) that, likewise, w’ < 0 associated with u’> 0.

Turbulence Fluxes

(Negatively Correlated, Typical of ABL)

Again, use $\overline{u'w'}$ as an example …



In this example,
I have drawn the u and w traces to reflect that most of time u’ and w’ are negatively correlated (i.e. u’ > 0 with w’ < 0, and u’ < 0 with w’ > 0).

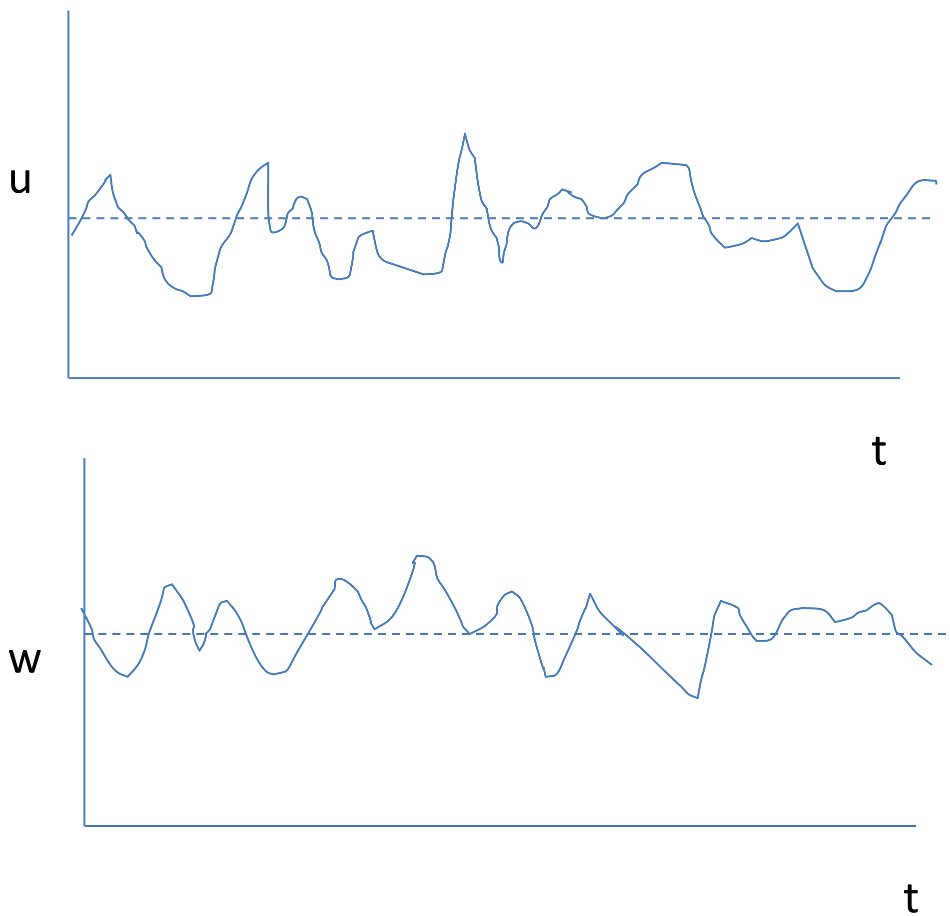
$$R_{uw} = \frac{\overline{u'w'}}{\sigma_u \sigma_w} < 0$$

“correlation coefficient”
For u’w’. In this case < 0.

Compare previous slides …

(Random example vs. negatively correlated example)

Negatively correlated …

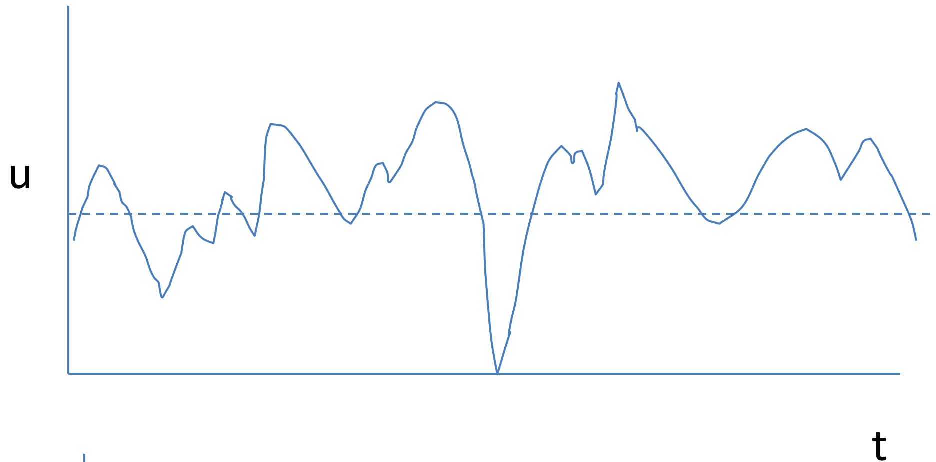


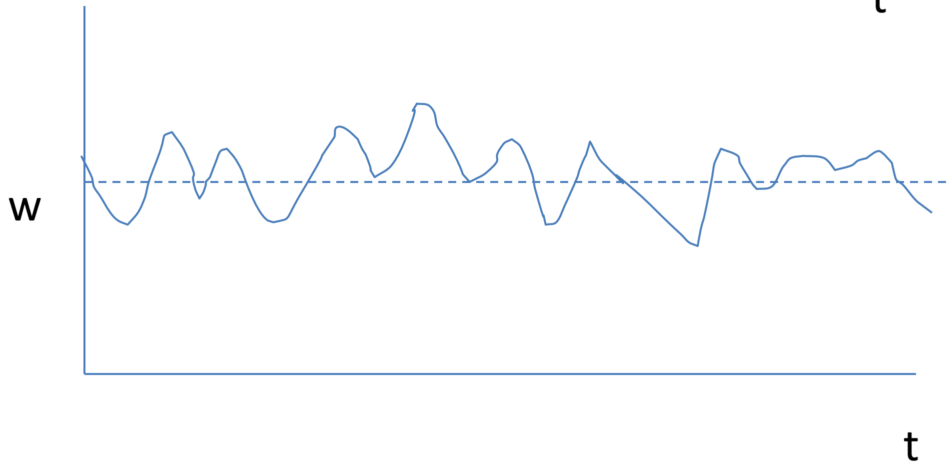
both “appear” random
(but they aren’t …)

Compare previous slides …

(Random example vs. negatively correlated example)

Random example …





both "appear" random
(and they are …)

Daytime:
Convective Boundary Layer (CBL)

“Fair-weather” cumulus
(Cumulis Humulis)

Cloudy regions indicate
Regions of “updrafts”
in CBL … moving moisture
upwards with eventual
condensation and cloud
formation.

free →
troposphere

mixed →
layer

surface →
layer

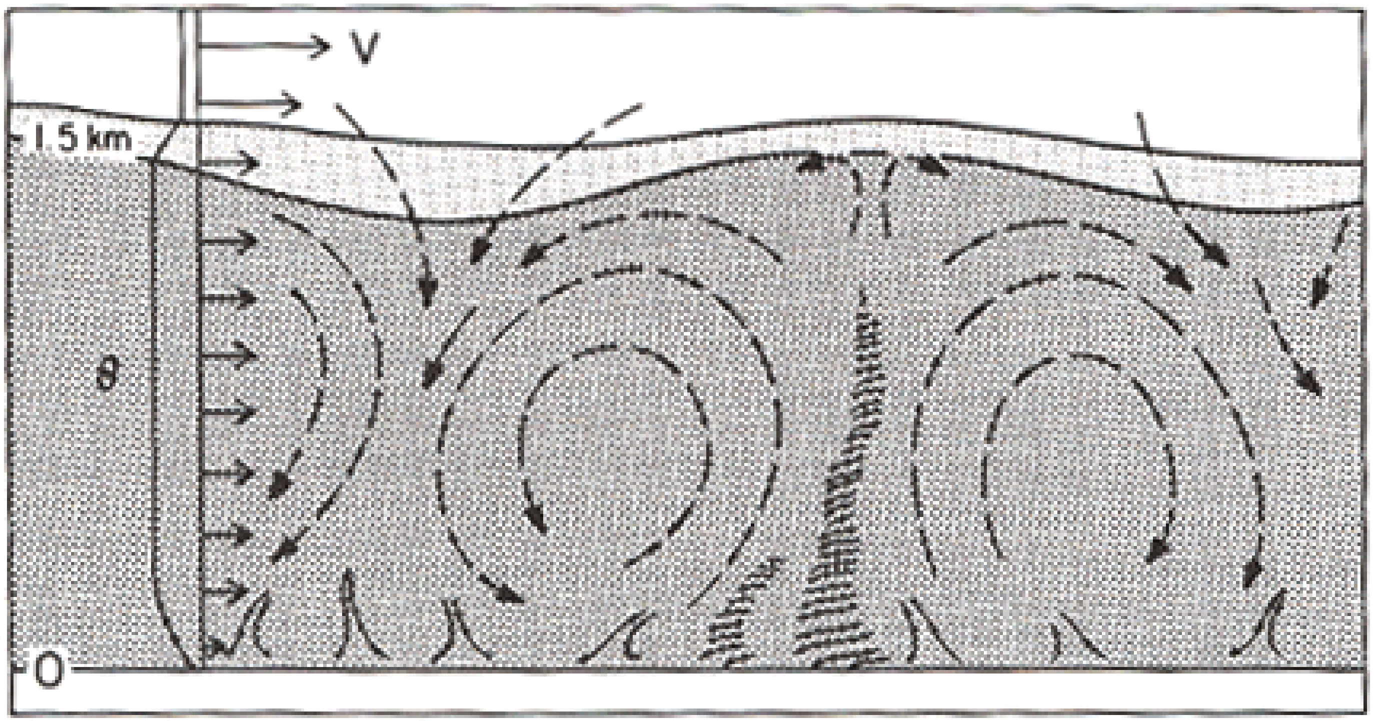
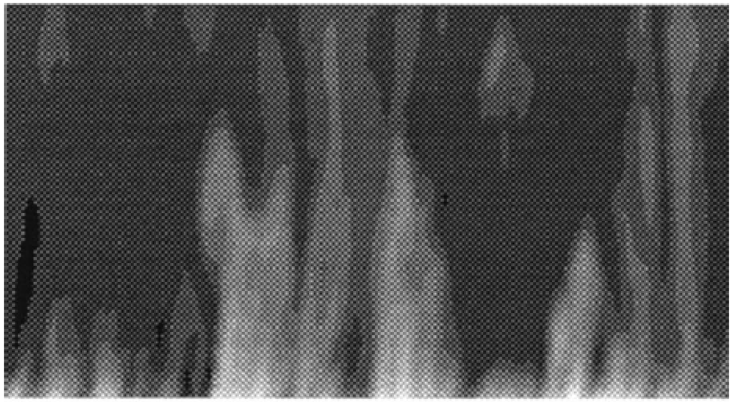


FIG. 1.4. Schematic of convective boundary layer circulation and entrainment of air through the capping inversion (from Wyngaard, 1990).

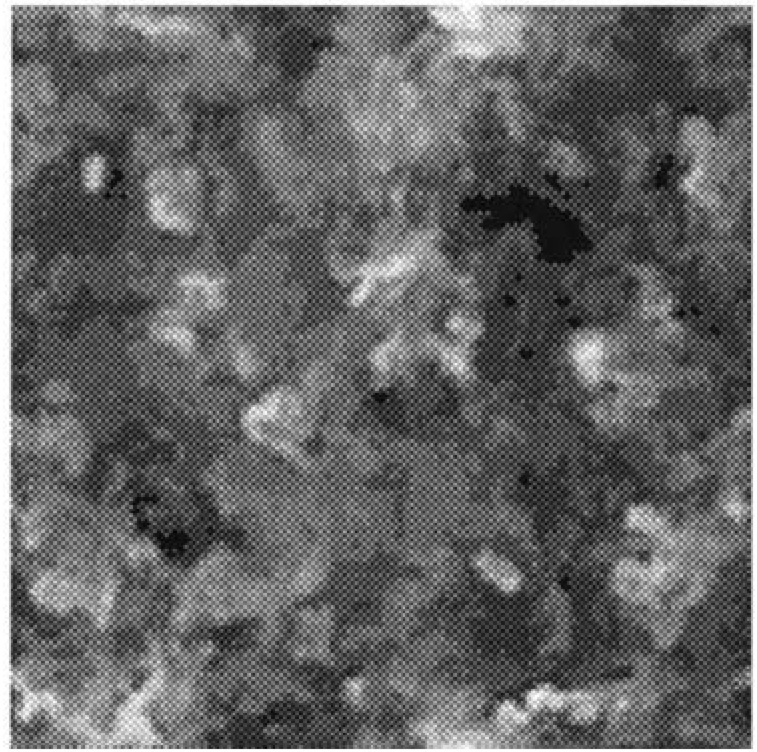
Convective Updrafts & Downdrafts

(Convective Boundary Layer generated from LES computer simulation)

vertical cross section



Horizontal cross section

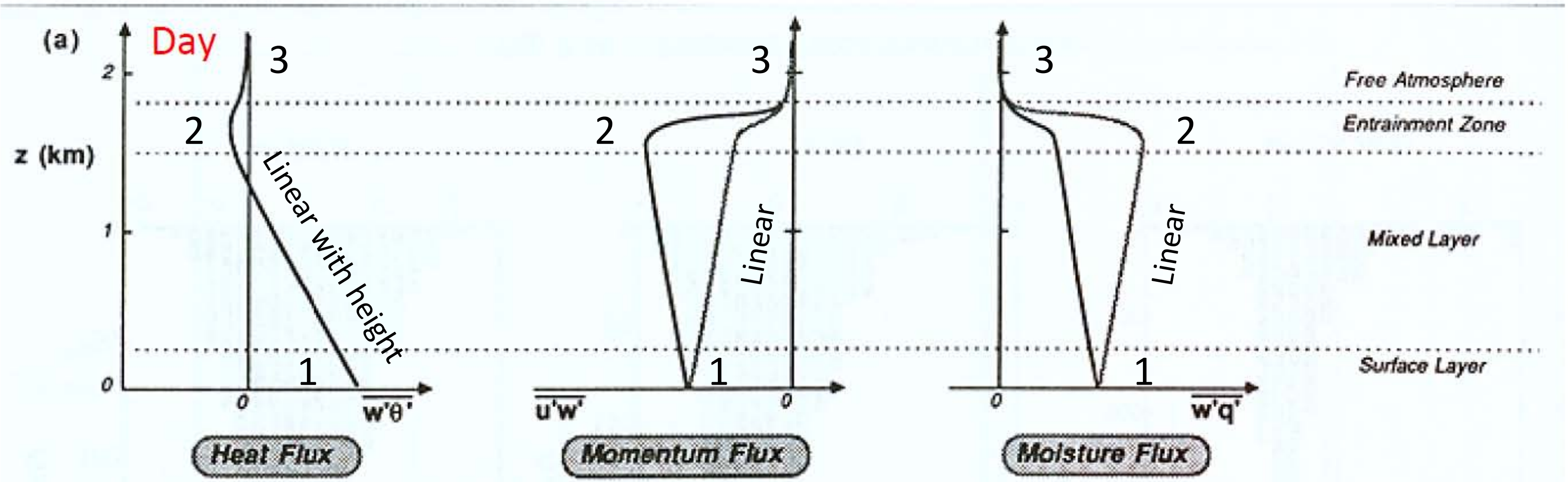


White: Updrafts
Grey and darker: Downdrafts

FIG. 1. A typical snapshot of the potential temperature field θ, in the quasi-steady state of a convective boundary layer simulated by a large eddy simulation with resolution 128^3. (top) Vertical cross section restricted to the mixed layer; (bottom) horizontal cross section inside the mixed layer. Gray shades are coded according to the intensity of the field: white corresponds to large temperature, black to small ones. Plumes and well-mixed regions are clearly detectable.

Typical flux profiles in the daytime ABL

(Stull Figure 2.15, two lines are two typical cases)



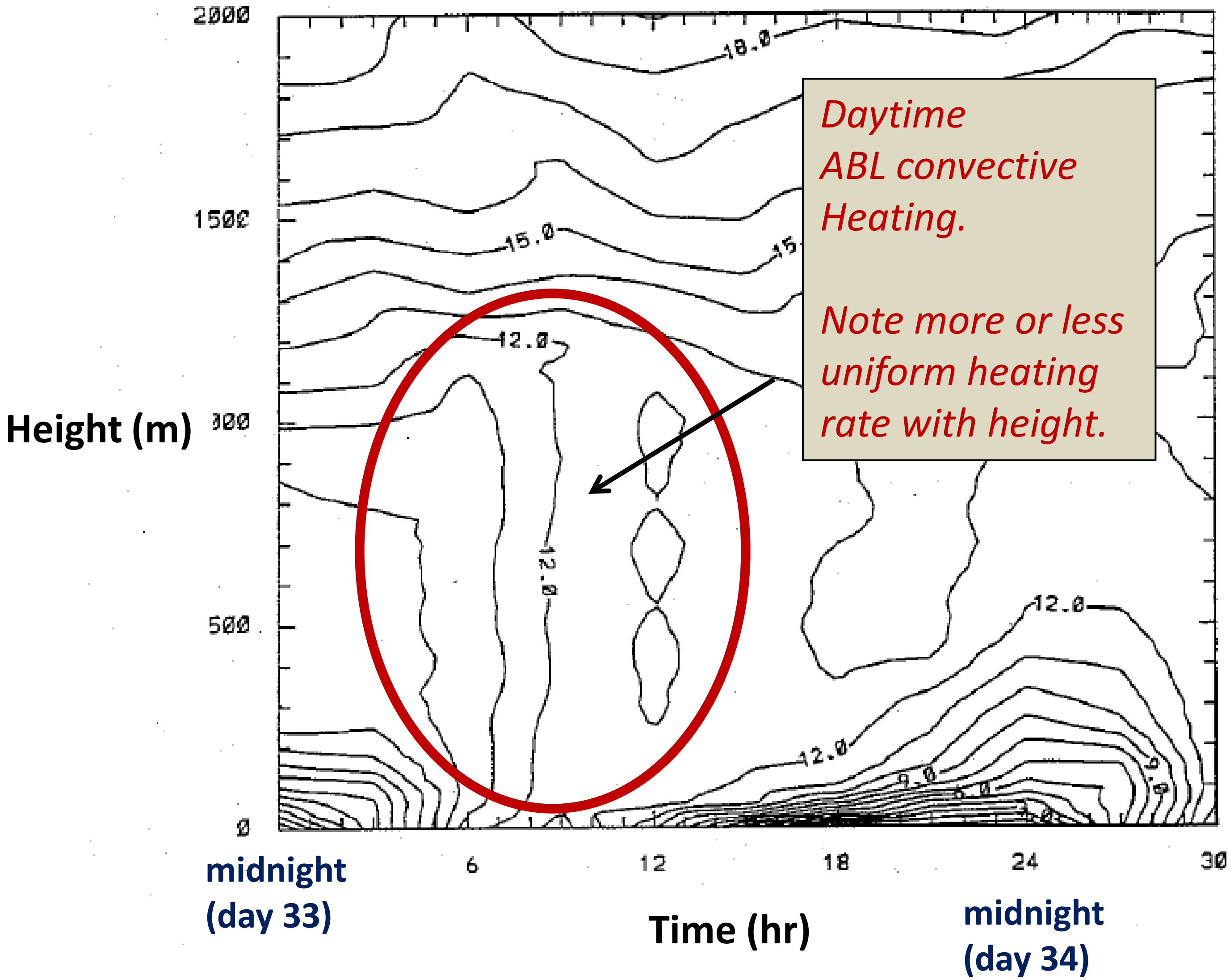
NOTE THREE POINTS ALONG FLUX PROFILES (see labeling above on far left) …

1. Surface flux (values above at z = 0)
2. Entrainment flux (value in middle of entrainment zone, point where profile breaks from linear)
3. Point where flux equals zero atop ABL

***** Profiles tend to be linear between points 1 and 2 *****

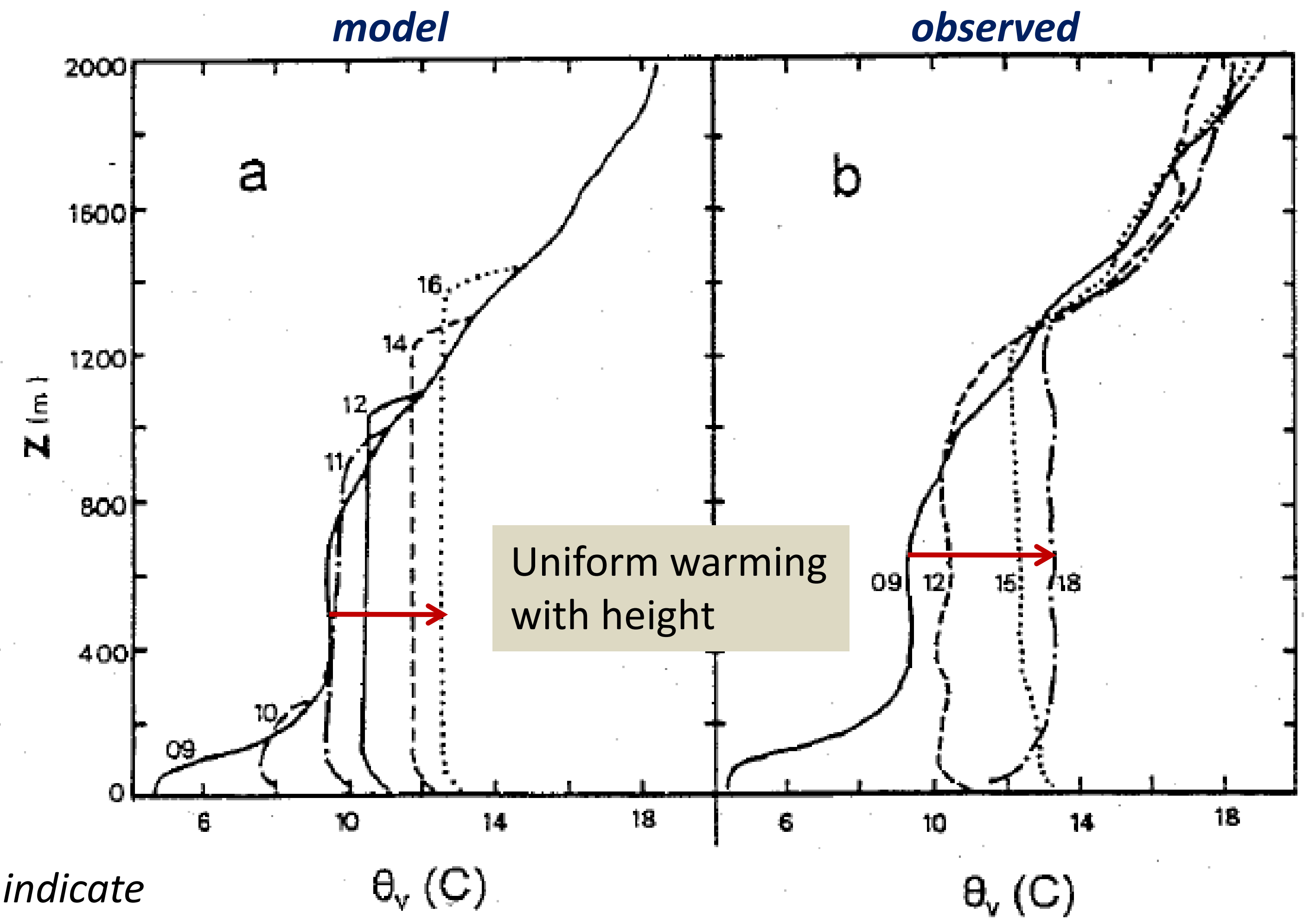
Diurnal Potential Temperature on Wangara Day 33

(classic ABL field experiment, Australia)



Mean Potential Temperature Profiles vs. Time

(Daytime ABL heating; Wangara Day 33)



Lines indicate hour of day

Explanation …

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial (\overline{w'\theta'})}{\partial z} \approx constant$$

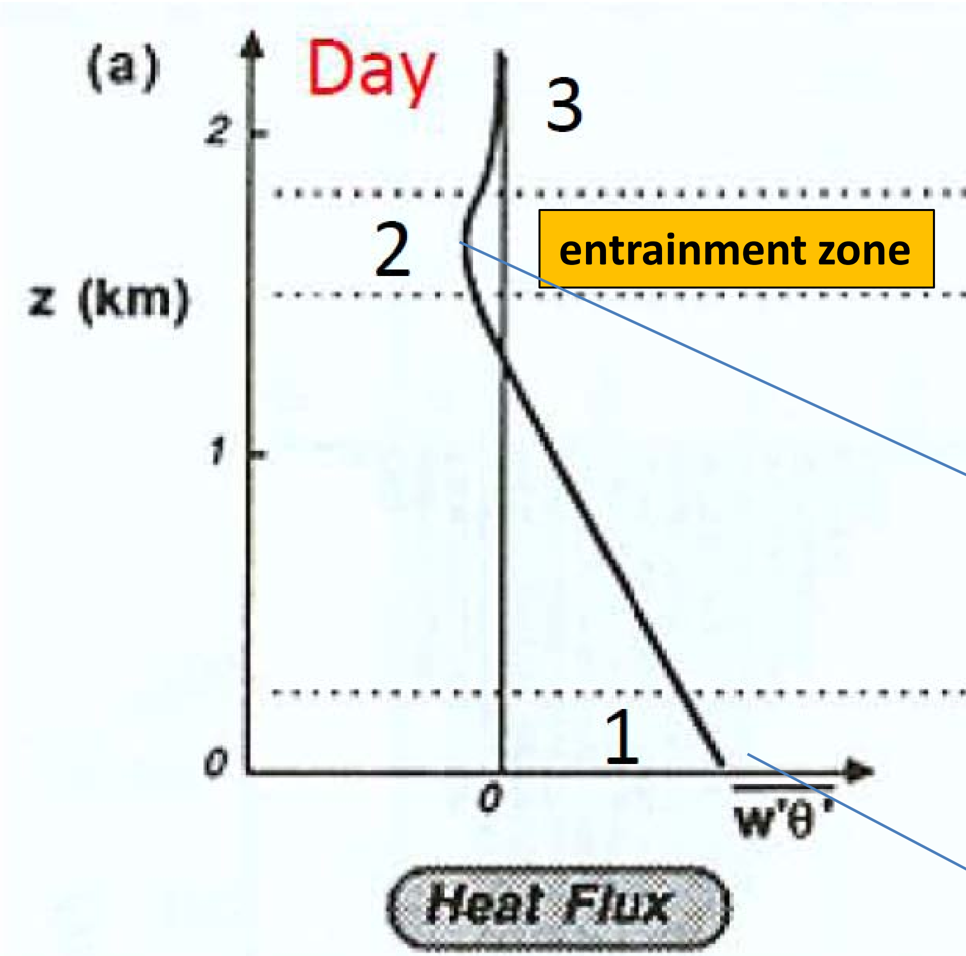
(… since flux varies ~ linearly with height within ABL)

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial (\overline{w'\theta'})}{\partial z} \approx -\frac{\left[(\overline{w'\theta'})_e - (w'\theta')_0\right]}{h} = -\frac{[(<0)]}{h} = (>0)$$

(… since flux <u>decreases</u> linearly with height,
Therefore flux-divergence is greater than zero.)

Result … warming rate within daytime ABL tends to be uniform with height.

Entrainment



Note: Point 2 is point where turbulent flux breaks from linear (inflection point in flux profile).

$\left(\overline{w'\theta'}\right)_e < 0$

Downward directed entrainment flux

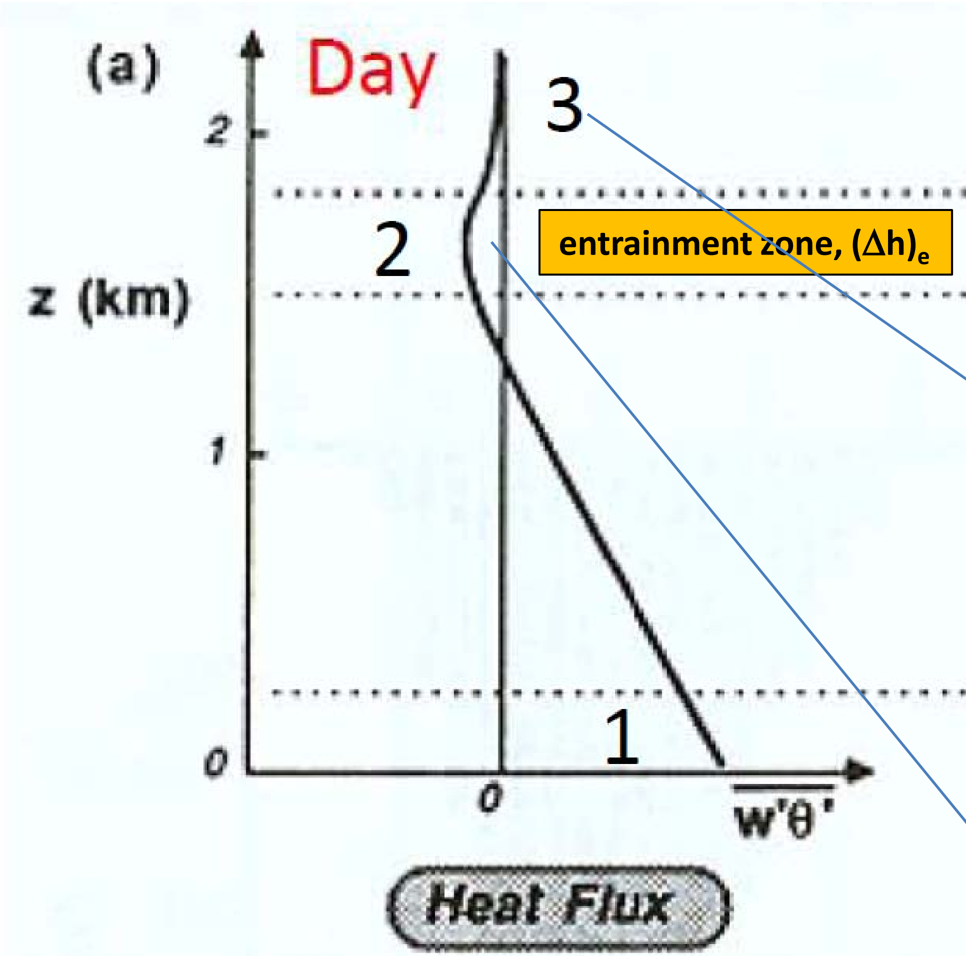
ABL **warming** from below and (more weakly) from above, the latter due to entrainment.

h

$\left(\overline{w'\theta'}\right)_0 > 0$

Upward directed Surface heat flux

But what about at top of entrainment zone?



Note: Point 2 is point where turbulent flux breaks from linear (inflection point in flux profile).

$$\left(\overline{w'\theta'}\right) = 0$$

Zero flux @ top of ABL

Cooling … as heat from entrainment zone being fluxed in ABL, and relatively cool air from ABL being fluxed into EZ. i.e. heat exchange & mixing between ABL and EZ.

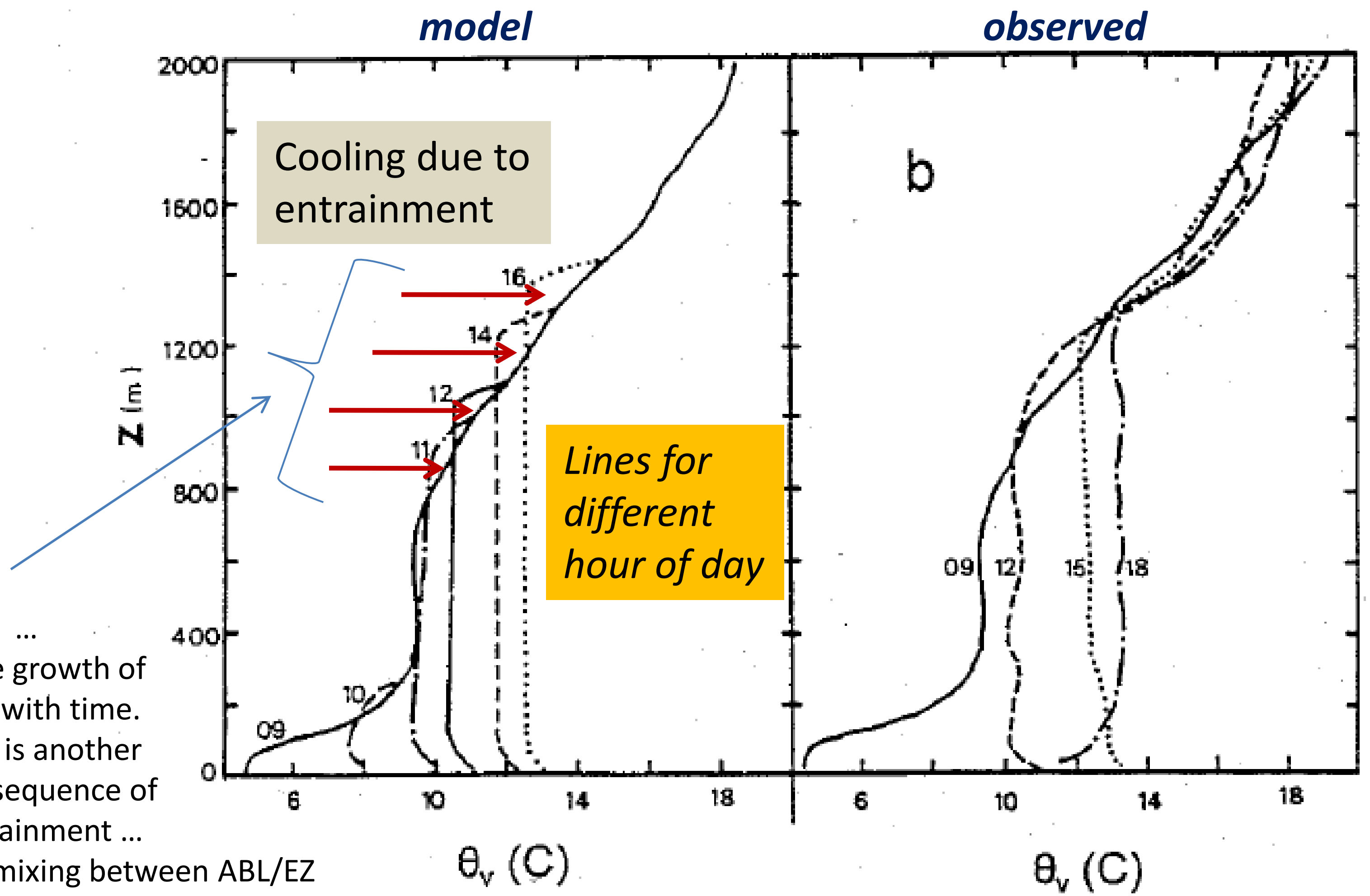
$(\Delta h)_e$

$$\left(\overline{w'\theta'}\right)_e < 0$$

Downward entrainment flux

Mean Potential Temperature Profiles vs. Time

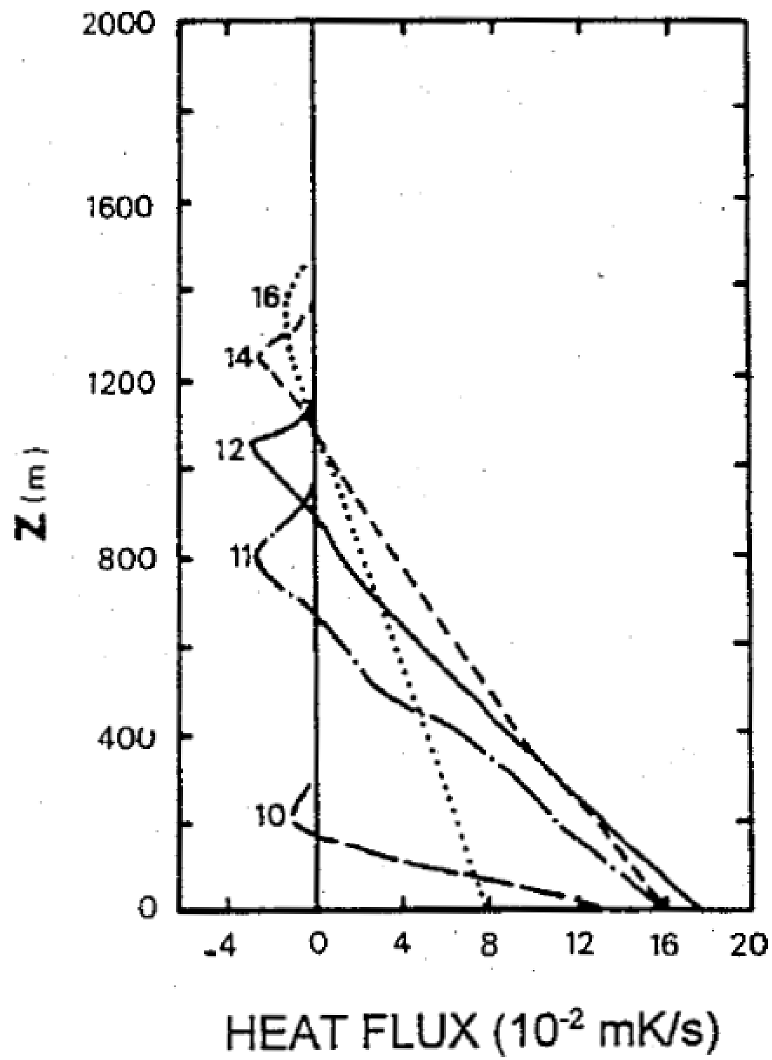
(Daytime ABL heating; Wangara Day 33)



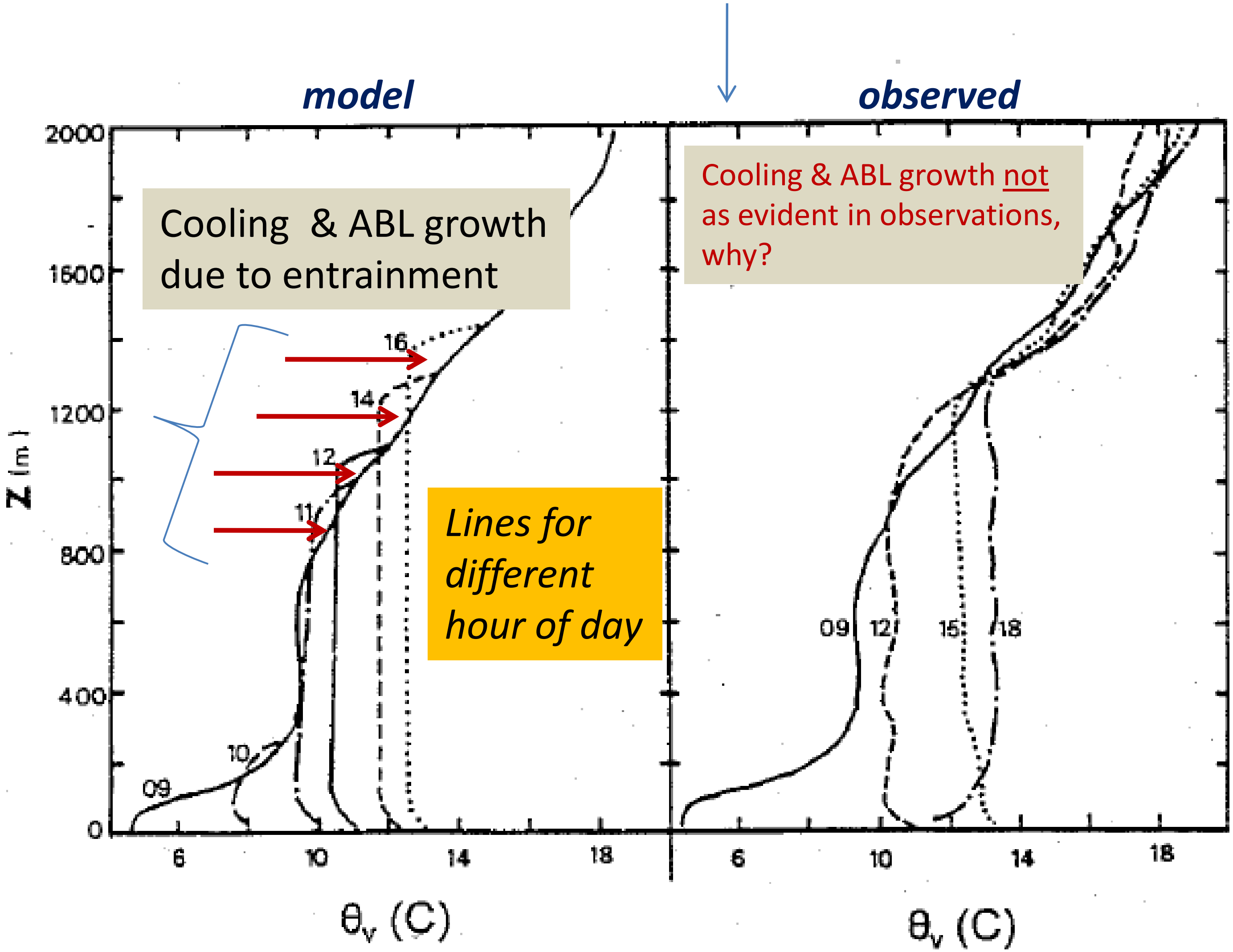
Also …
note growth of ABL with time.
This is another consequence of entrainment …
i.e. mixing between ABL/EZ brings θ profile in this region quasi-neutral (constant θ with z)

Corresponding Heat Flux ($\overline{w'\theta'}$) Profiles vs. Time

(Note ABL growth in time … but flux profile still has same basic shape)



But what about observations? Right side of plot …



ABL Growth Rate (1)

(daytime ABL)

Can be shown, assuming linear flux profiles in ABL , that …

$$\frac{\partial h}{\partial t} = w_e + w_{sub}$$

where $$w_e \equiv \frac{-\left(\overline{w'\theta'}\right)_e}{\left(\Delta\overline{\theta}\right)_e}$$

w_e is termed the "entrainment velocity", with $(\Delta\theta)_e = \theta_h - \theta_{abl}$ the mean potential temperature "jump" from bottom to top of entrainment zone.

and w_{sub} is the large-scale (synoptic, general circulation) mean vertical velocity (called w_{sub} because often < 0 due to large-scale subsidence)

ABL Growth Rate (2)

(daytime ABL)

$$\frac{\partial h}{\partial t} = w_e + w_{sub}$$

$$w_e \equiv \frac{-\left(\overline{w'\theta'}\right)_e}{\left(\Delta\overline{\theta}\right)_e} = \frac{-[<0]}{[>0]} = [>0]$$

Entrainment velocity > 0.
Leads to ABL growth, as expected.

w_{sub} on the other hand is <u>negative</u> during large-scale subsidence. (Fair-weather, synoptic scale high pressure situation).

Therefore we and wsub often counter each other. Daytime ABL growth therefore often capped as a result of subsidence.

Synoptic Scale Vertical Velocity

(Stull Figure 1.6)

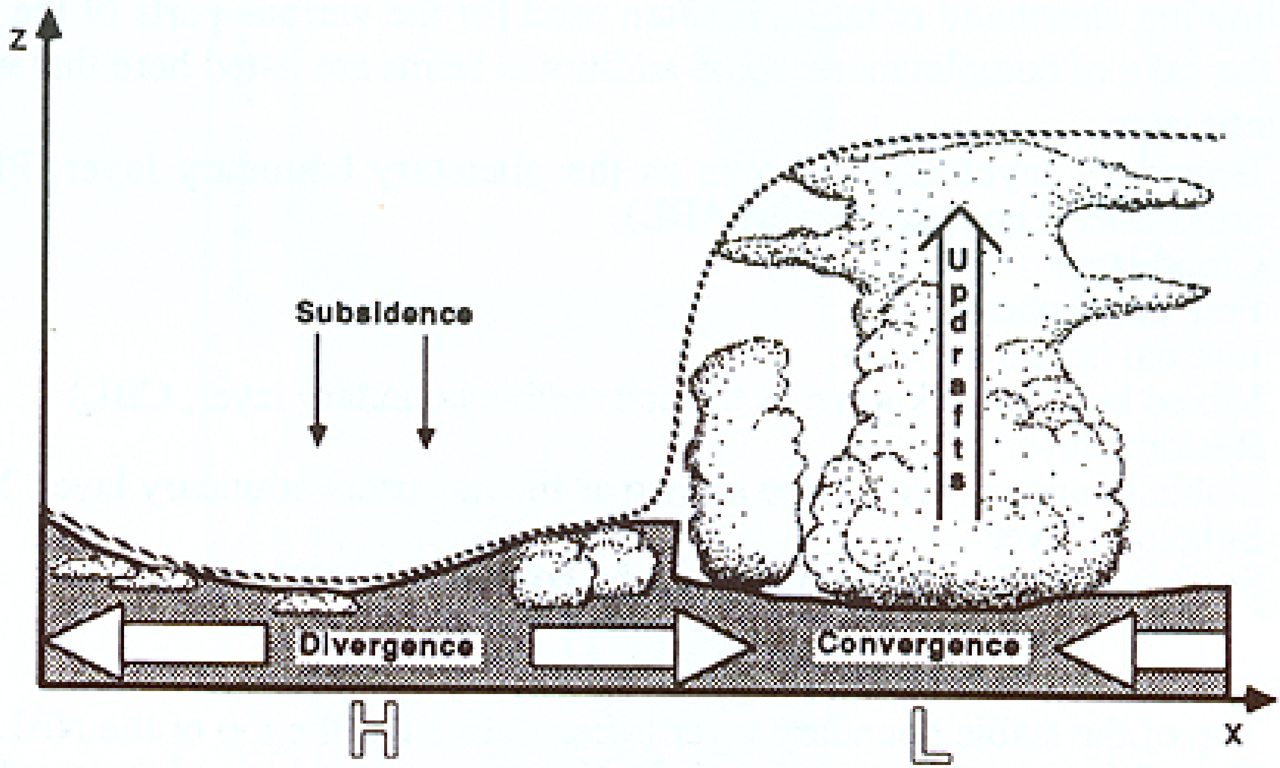
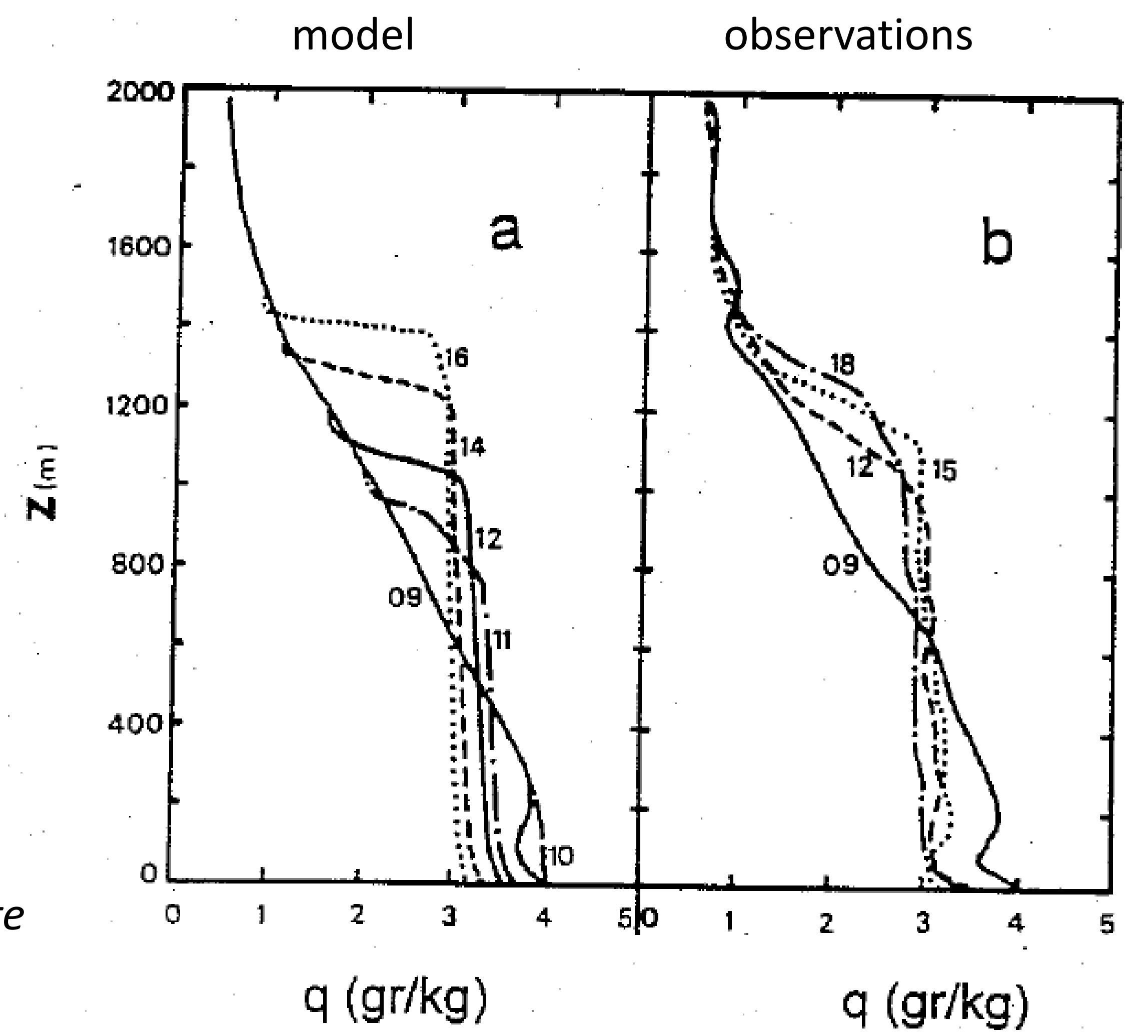


Fig. 1.6 Schematic of synoptic - scale variation of boundary layer depth between centers of surface high (H) and low (L) pressure. The dotted line shows the maximum height reached by surface modified air during a one-hour period. The solid line encloses the shaded region, which is most studied by boudary-layer meteorlogists.

from Stull 1988

Mean Specific Humidity Profiles vs. Time

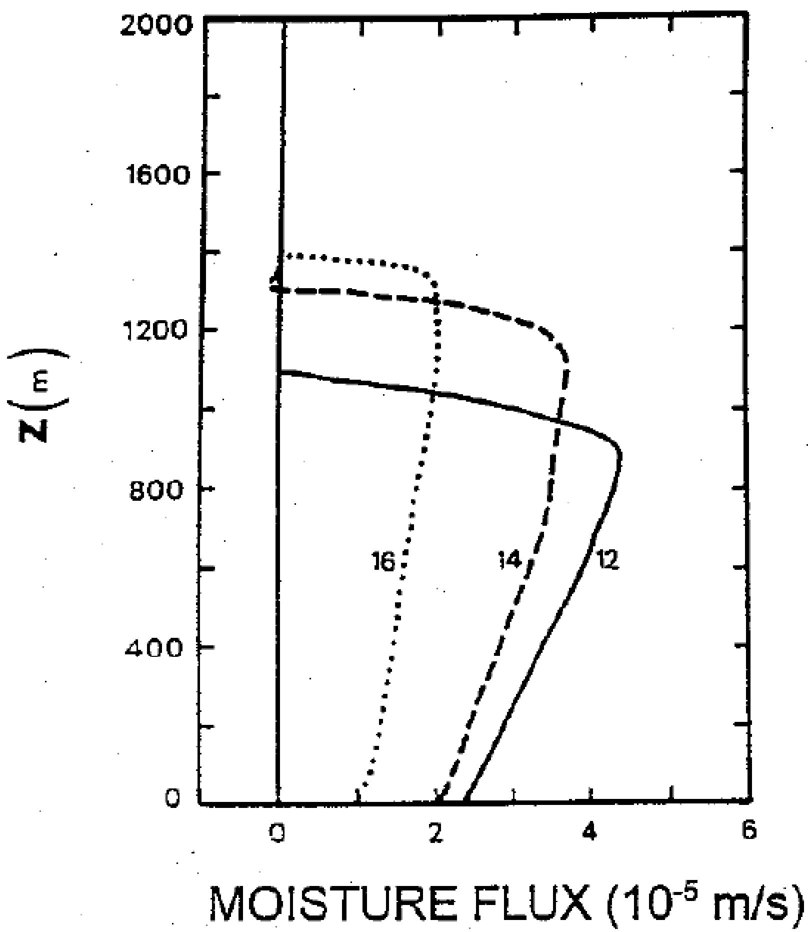
(Daytime ABL; Wangara Day 33)



Lines indicate hour of day

Corresponding Moisture Flux (w'q') Profiles vs. Time

(Note ABL growth in time … but flux profile still has same basic shape)



Nighttime:
Stable Boundary Layer (SBL)

Before we start

(rate equation for Turbulent Kinetic Energy, TKE)

Let E = turbulent kinetic energy = $\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)/2$

Then, a rate equation for TKE can be derived …

$$\frac{\partial E}{\partial t} = -\overline{u'w'}\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z} + \frac{g}{\theta_a}\overline{w'\theta'} - \varepsilon + \frac{\partial}{\partial z}\left[K_m \frac{\partial E}{\partial z}\right]$$

Shear production

buoyancy production (or destruction)

molecular dissipation

Vertical turbulent diffusion (“transport”)

Rewritten using K-theory …

$$\frac{\partial E}{\partial t} = K_m \left[\left(\frac{\partial \overline{u}}{\partial z} \right)^2 - \left(\frac{\partial \overline{v}}{\partial z} \right)^2 \right] - \frac{g}{\theta_a} K_h \frac{\partial \overline{\theta}}{\partial z} - \varepsilon + \frac{\partial}{\partial z} \left[K_m \frac{\partial E}{\partial z} \right]$$

Shear production

buoyancy production (or destruction)

molecular dissipation

Vertical turbulent diffusion (“transport”)

Shear production

- Positive
- Generates turbulence along direction of mean wind (i.e. u’2 and v’2, not w’2)
- “mechanically” driven turbulence

Buoyancy production (or destruction)

- Positive or negative (depending on stability)
- Generates (or destroys) turbulence along vertical component (w’2)
- “buoyantly” driven turbulence (or suppressed)

Stable Boundary Layer Schematic … notice turbulent eddies are more horizontally oriented than vertical. A consequence of stable stratification (buoyant destruction of TKE) inhibiting vertical turbulent kinetic energy, and therefore vertical length of eddies. Compare with corresponding picture for daytime boundary layer … in daytime BL vertical turbulence is enhanced, therefore eddies are large and vertically encompass enture boundary layer.

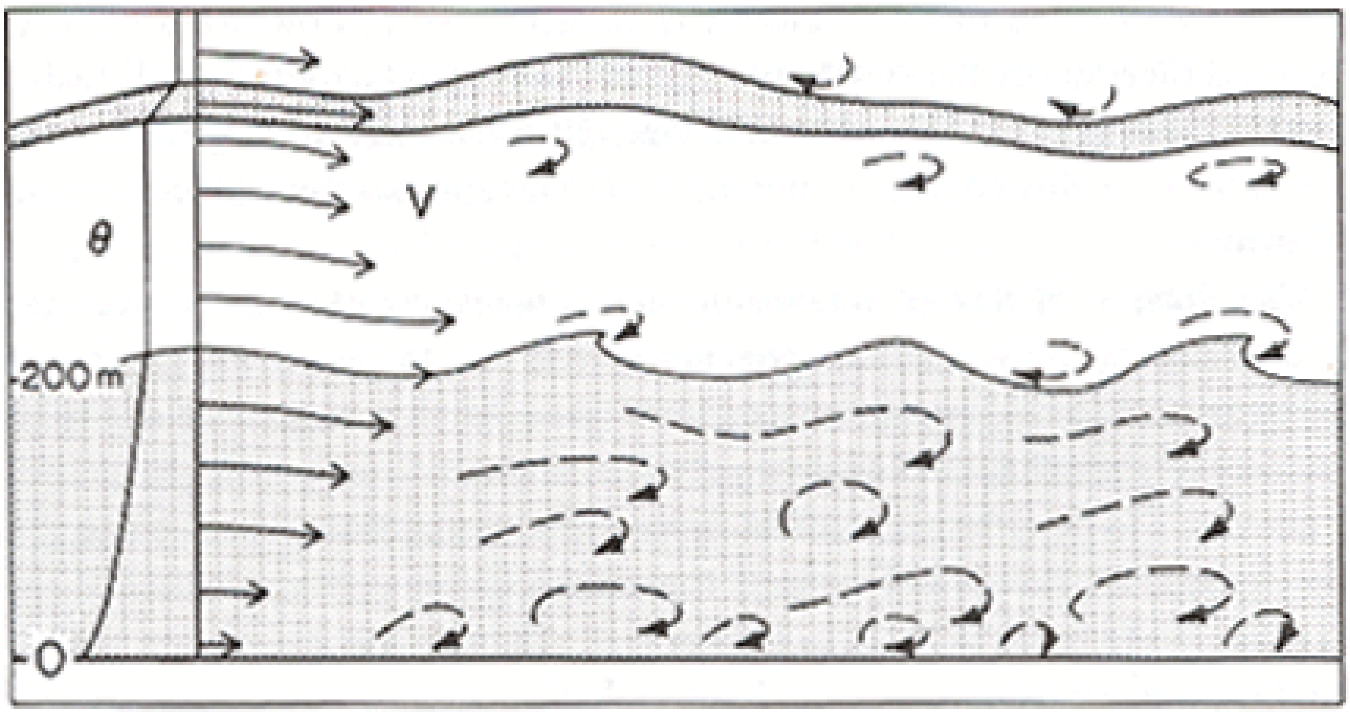


FIG. 1.5. Schematic of stable boundary layer flow showing eddy structure, waves, and elevated inversion layer (from Wyngaard, 1990).

Richardson Number (Ri)

- "Flux" Richardson Number (Ri_f)

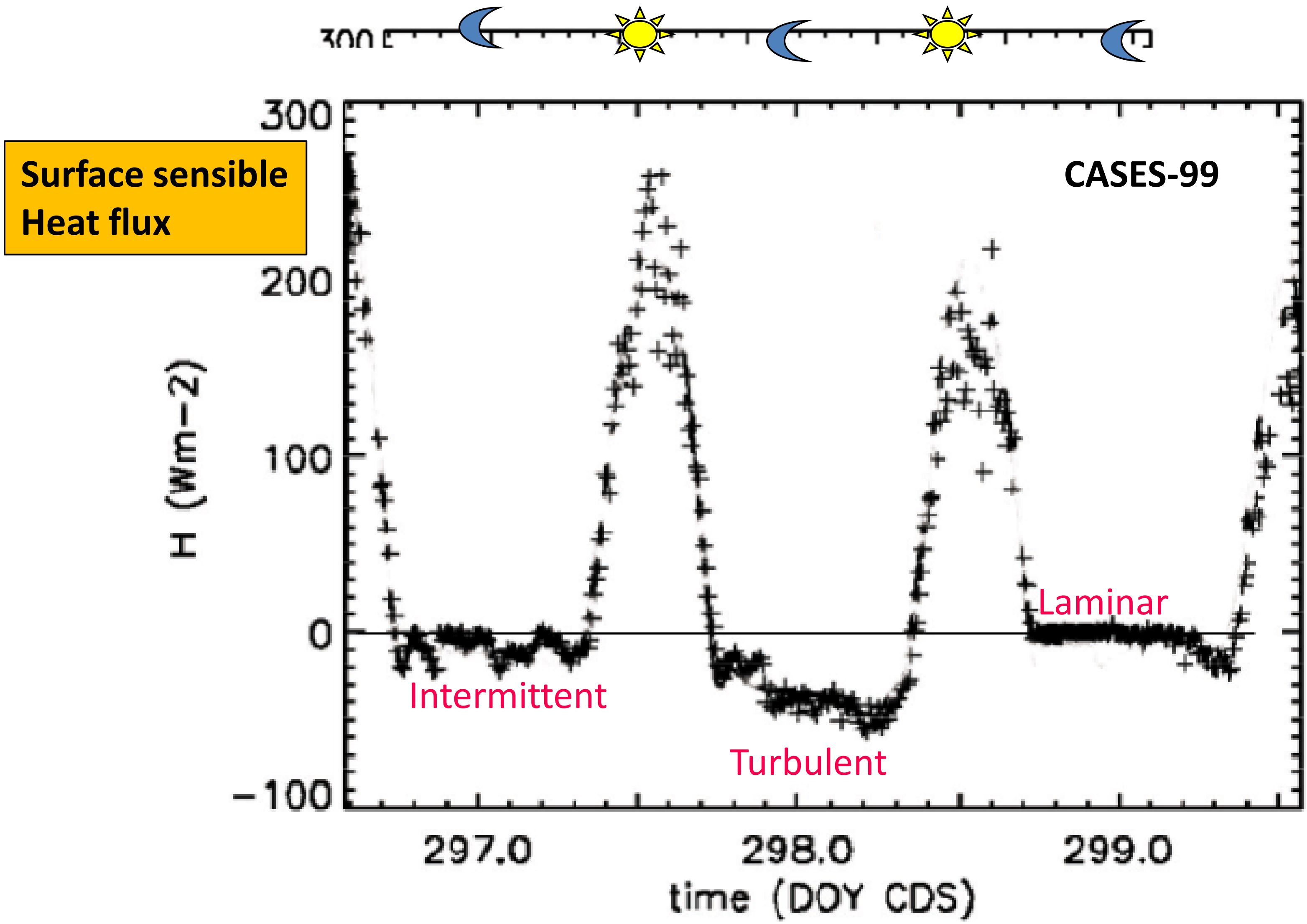
$$Ri_f = \frac{Buoyancy\ Destruction\ of\ TKE}{Shear\ Production\ of\ TKE} = \frac{K_H}{K_M} \frac{g/\theta_a \left(\partial\overline{\theta}/\partial z\right)}{\left(\partial\overline{u}/\partial z\right)^2}$$

- "Gradient" Richardson Number (Ri_g)

$$Ri_g = \frac{g/\theta_a \left(\partial\overline{\theta}/\partial z\right)}{\left(\partial\overline{u}/\partial z\right)^2} = \frac{K_M}{K_H} Ri_f = \Pr_t Ri_f$$

“Critical” Richardson Number (Ri_c)...

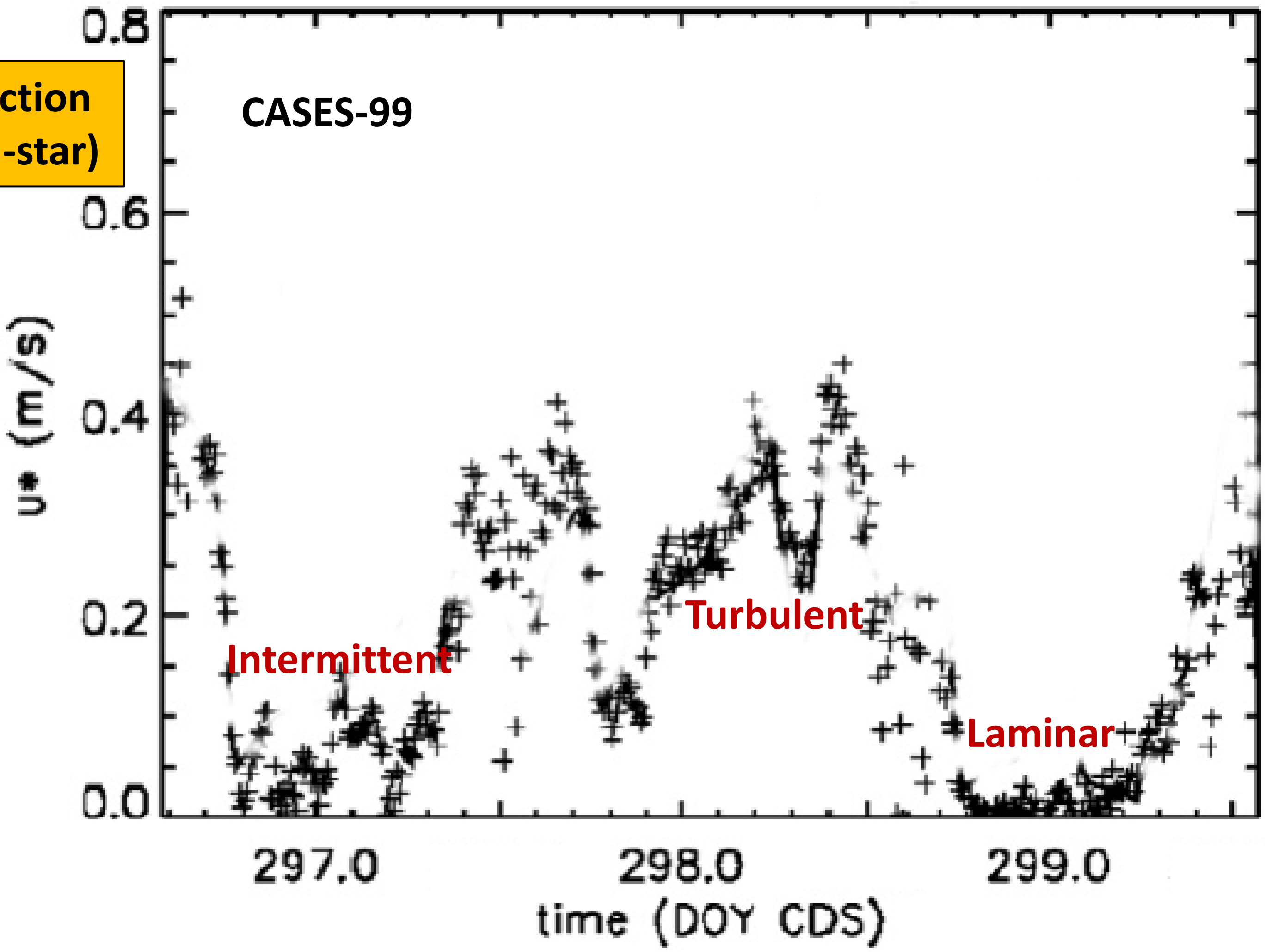
- A “critical” Richardson number exists in which turbulence generation cannot be sustained.
- That is … buoyant suppression of TKE is sufficiently strong to offset shear production
- Has been shown theoretically and experimentally … **$Ri_c \approx 0.25$** (=1/4).
- $Ri < \text{or} > Ri_c$ in stable boundary layer is a likely divider between continuously turbulent (“turbulent”, $Ri < Ri_c$) and intermittently turbulent or non-turbulent (“intermittent” or “laminar”, $Ri > Ri_c$) stable boundary layers observed in nature.

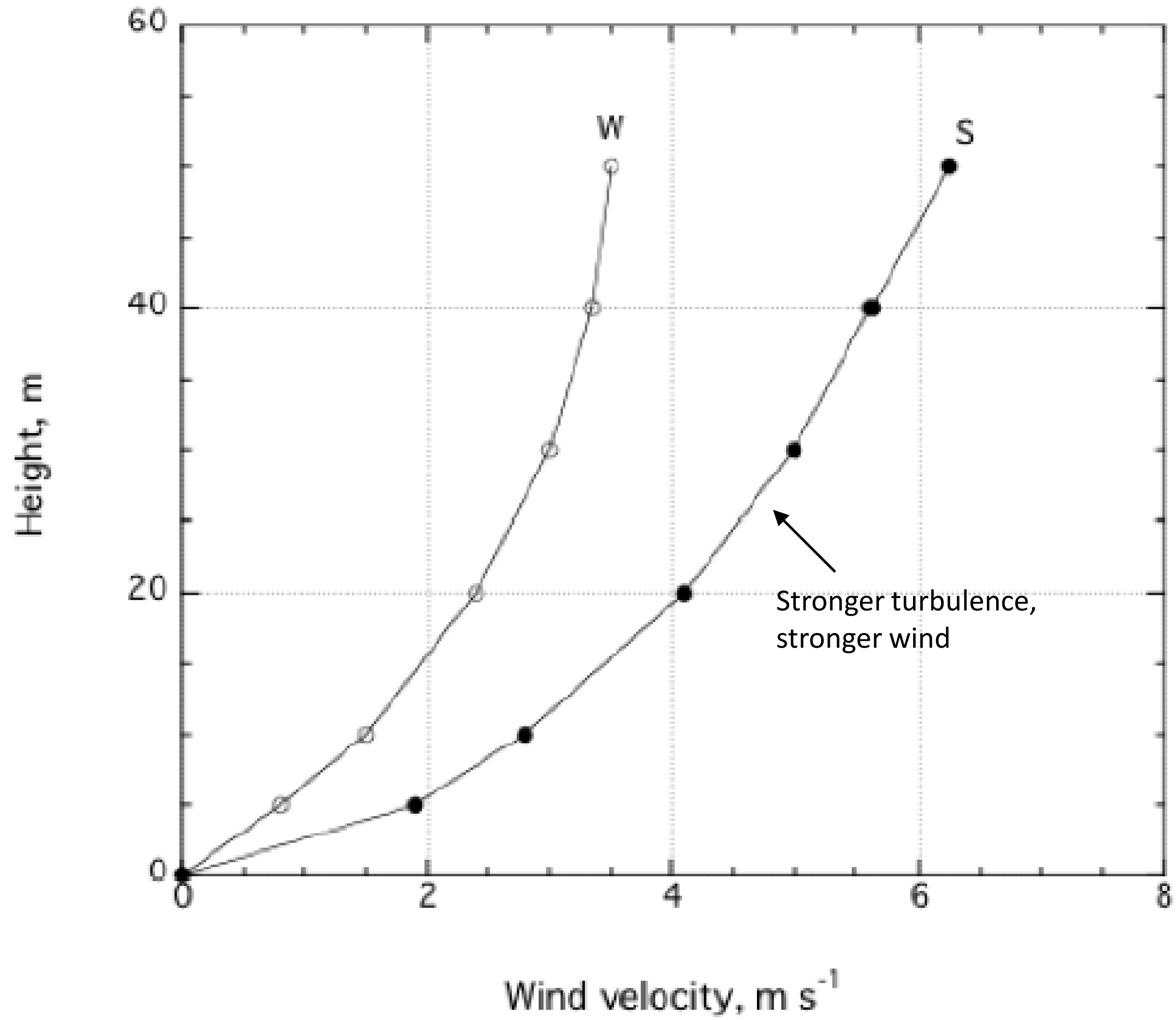


Three different days during CASES-99 experiment (Kansas-Oklahoma)

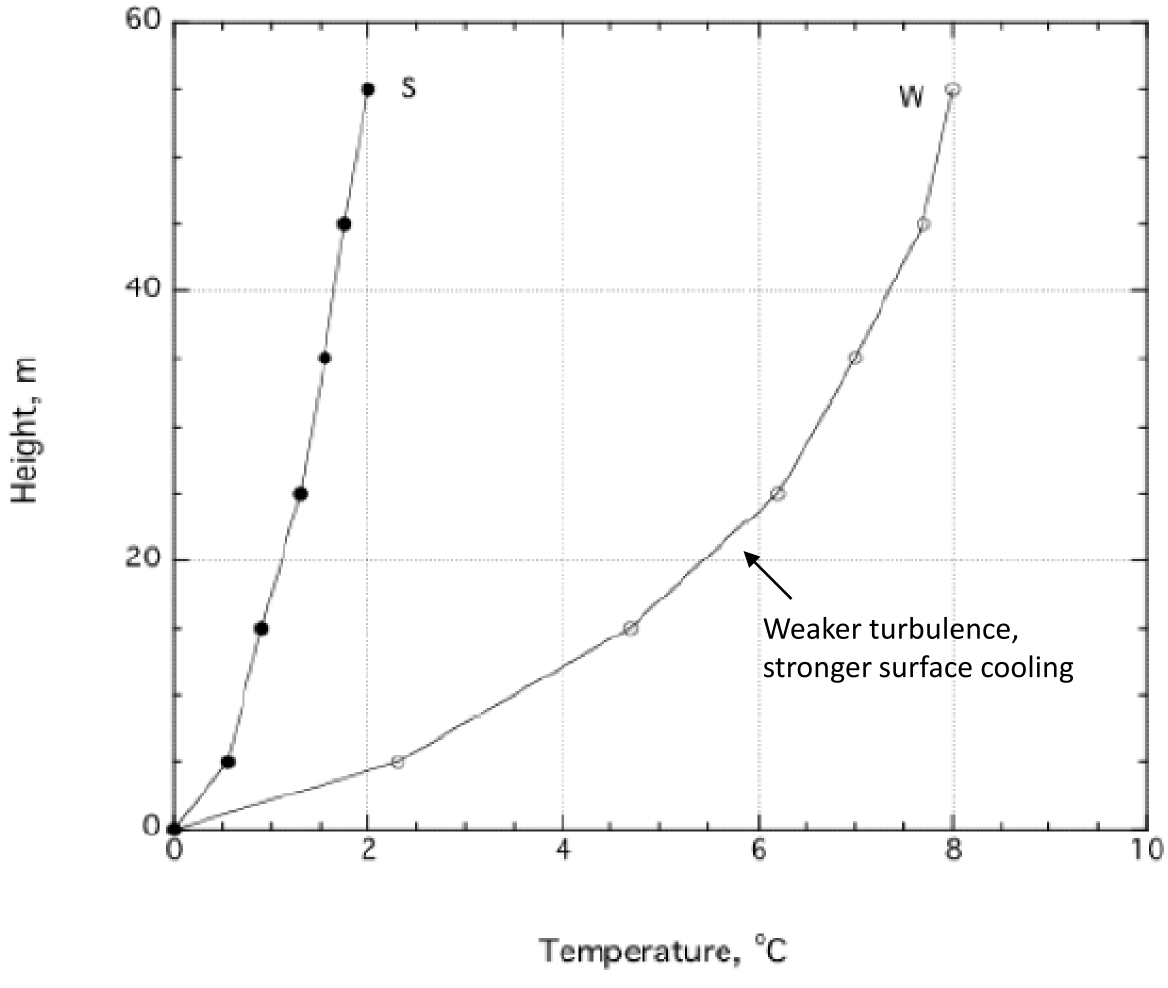
After Steeneveld et al

Surface friction velocity (u-star)

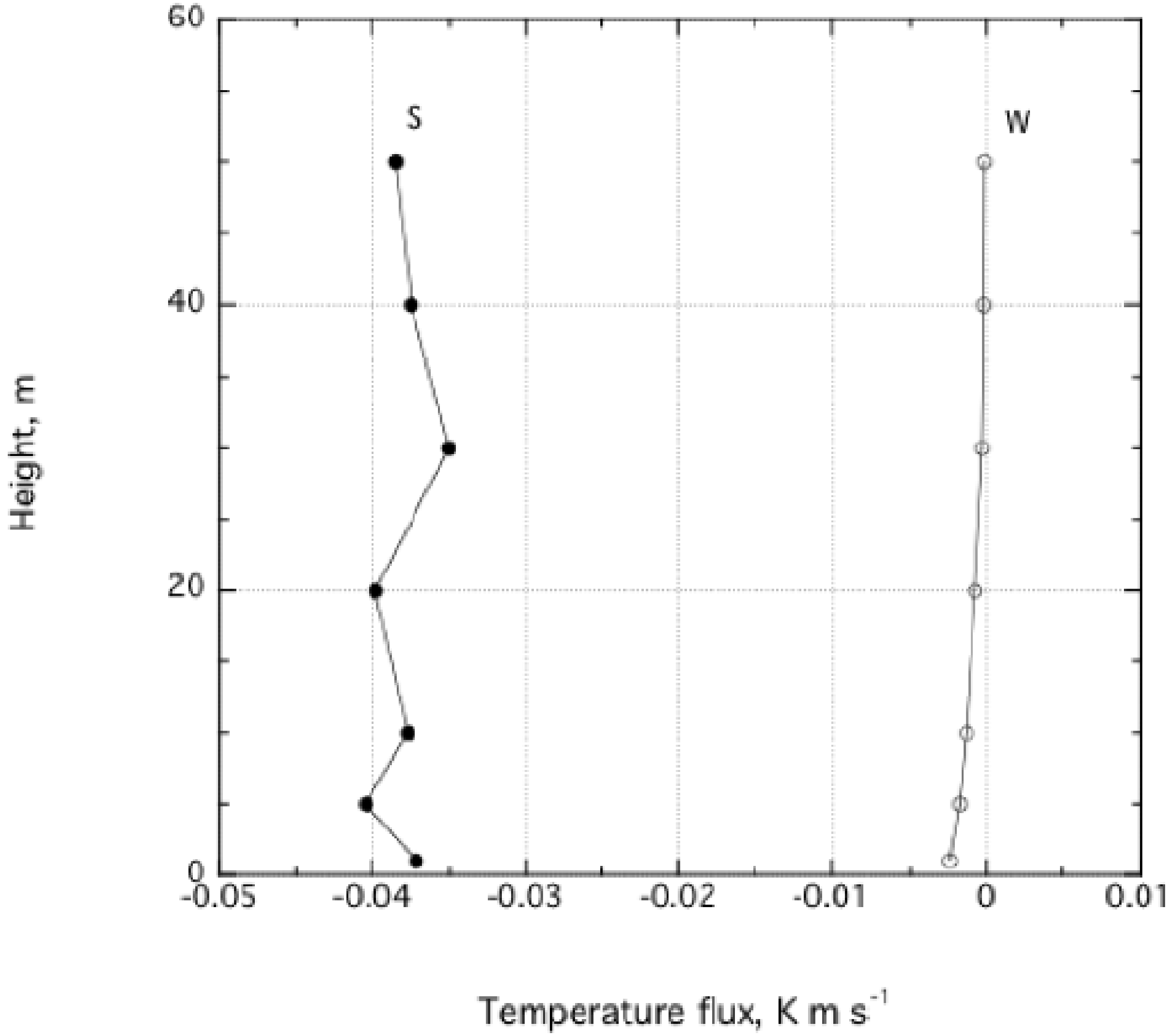




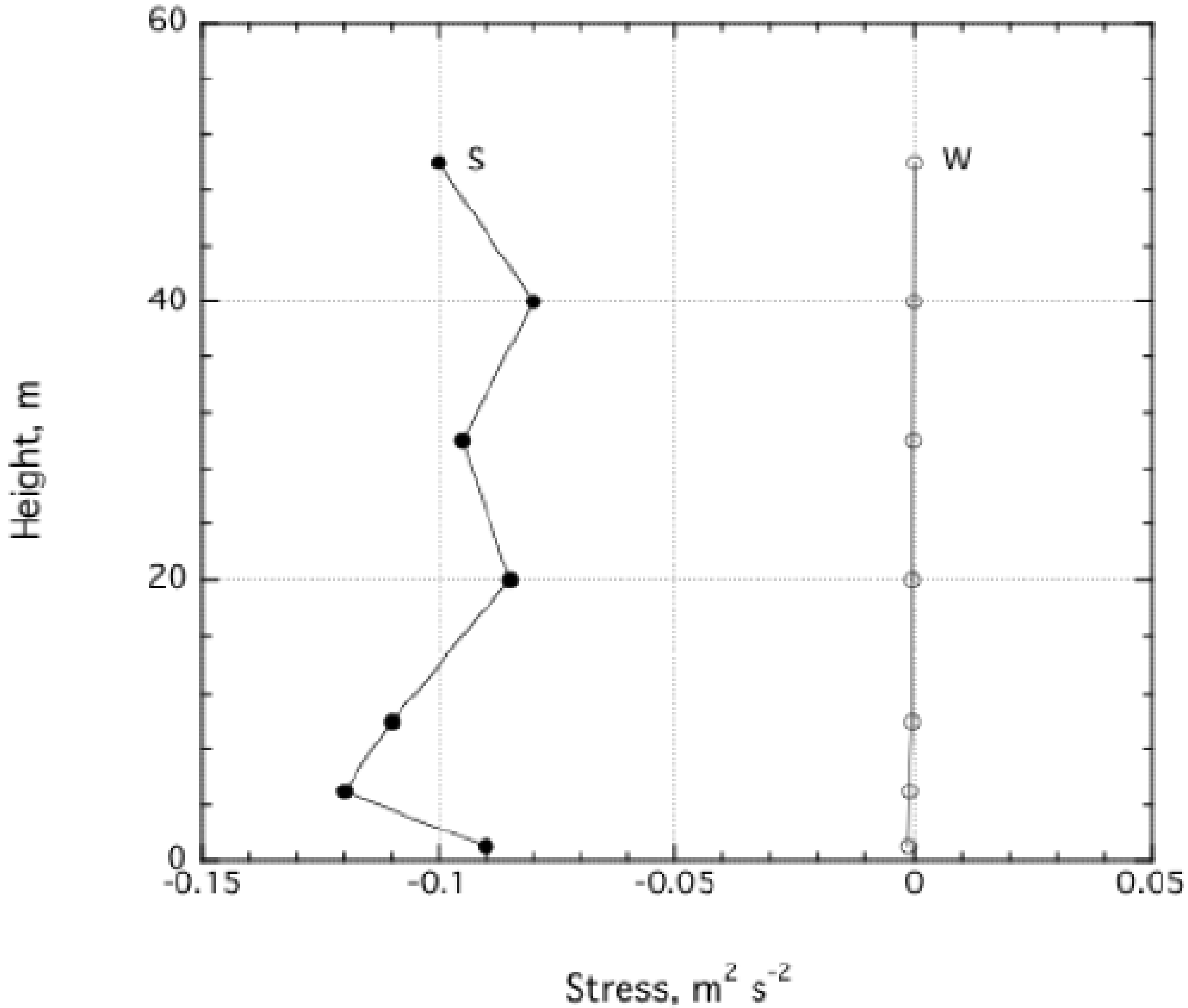
- Profiles of the wind velocity, in case W (open circles) and case S (filled circles).



- Profiles of: the potential temperature, in the composite case W (open circles) and S (filled circles). The potential temperature is the deviation from the surface value.

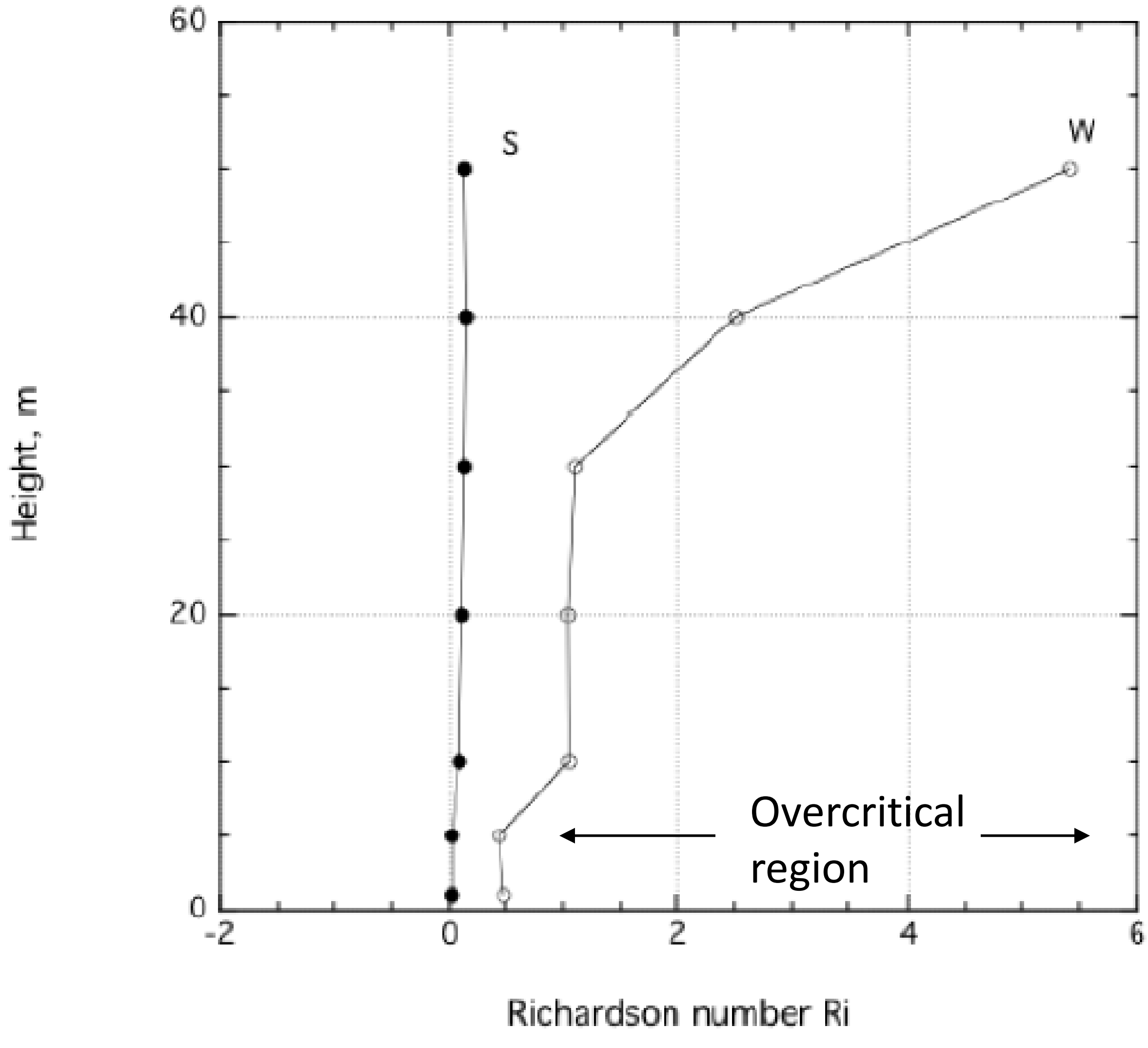


- Profiles of the temperature flux, in case W (open circles) and case S (filled circles).



100x

Profiles of the Reynolds stress, in case W (open circles) and case S (filled circles).



• Profiles of the **Richardson number *Ri,*** in case W (open circles) and case S (filled circles).

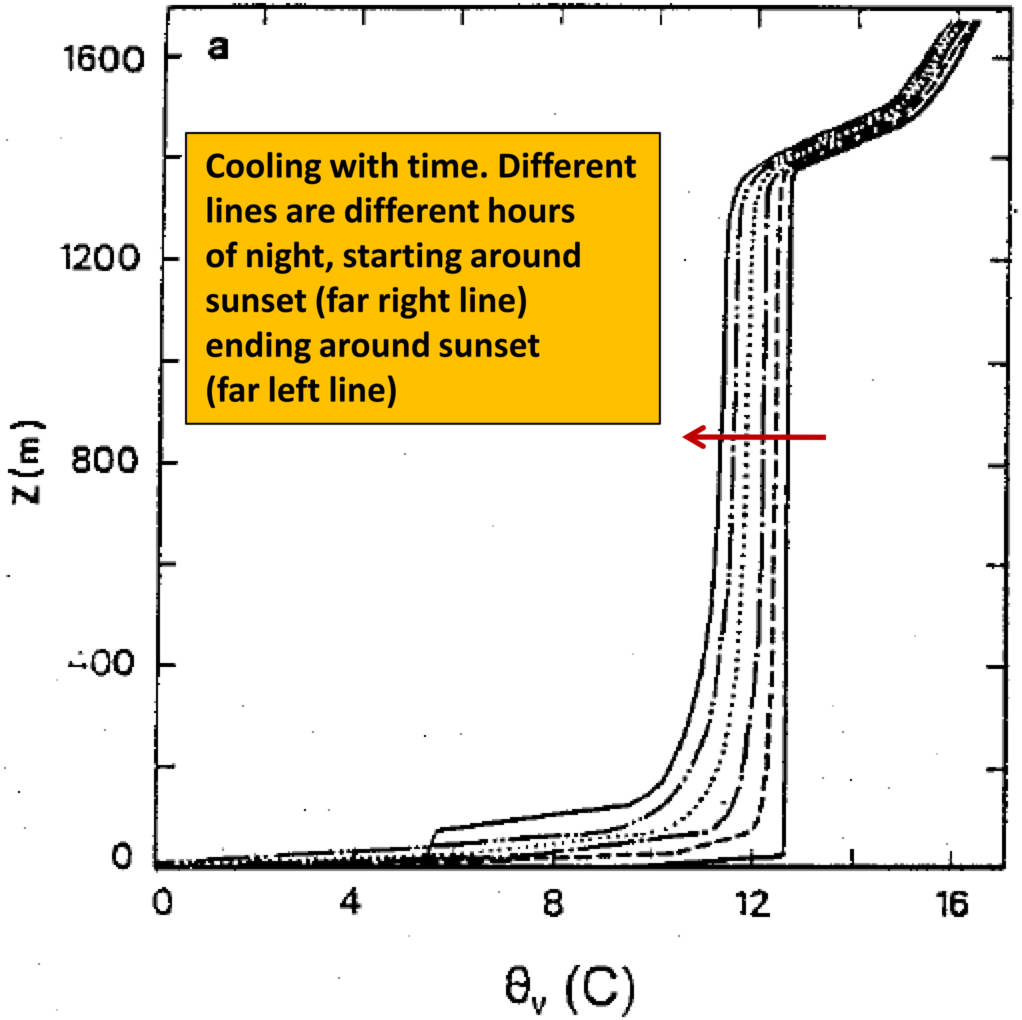
Focus on turbulent stable boundary layer
(Ri < Ri_c throughout most stable boundary layer)

Two issues will be investigated …

- Turbulence vs. Radiation in potential temperature profile
- Nocturnal (“low-level”) jet development in wind speed profile

Mean Potential Temperature Evolution

(Wangara Day 33 simulation)

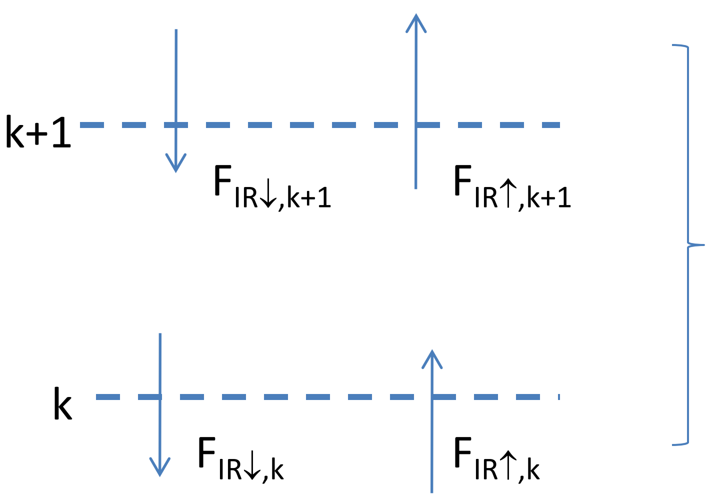


Potential Temperature
(Stable Boundary Layer)

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial(\overline{w'\theta'})}{\partial z} - \frac{\partial F_{RAD}}{\partial z}$$

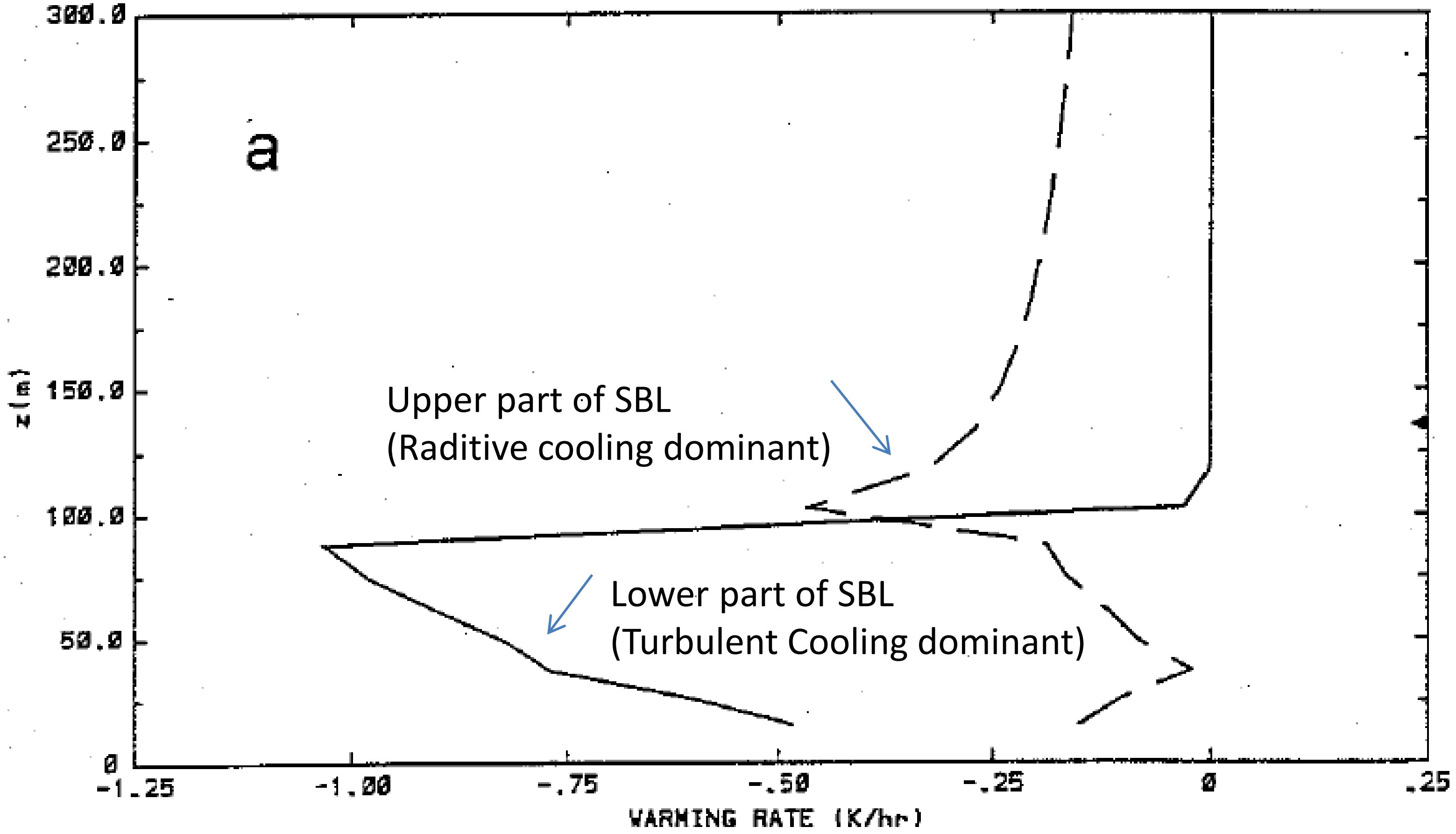
Vertical divergence of net IR radiative flux, F_{RAD}

$F_{RAD} = F_{IR\uparrow} - F_{IR\downarrow}$ (Upward minus downward IR flux)



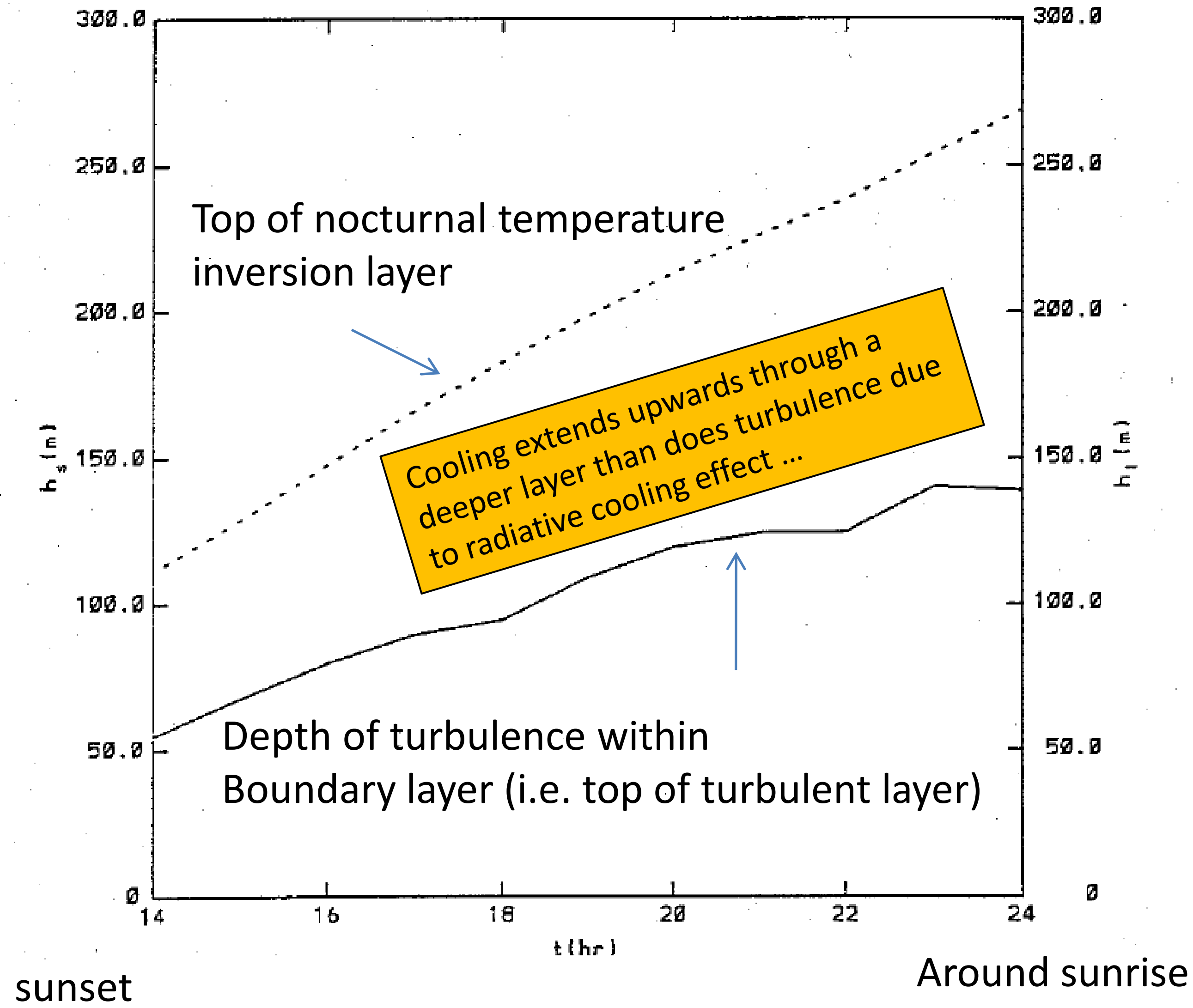
Upward and downward ir radiative flux across two vertical levels k & k+1. Divergence (convergence) of these fluxes leads to radiative cooling (warming) of this layer. This is an important process in the understanding cooling profiles in the nighttime, stable boundary layer over land.

Turbulent (solid) vs. Radiative (dashed) cooling



Stable Boundary Layer Depth

(Wangara Day 33 simulation)



Nocturnal Jet

(From Garratt Chapter 6.2.7)

Basic Explanation

- Abrupt decrease in turbulence in ABL during transition from daytime to nighttime (due to switch from unstable to stable conditions).
- Turbulent flux divergence in upper SBL (and RL) becomes practically zero.
- Wind accelerates (and rotates) towards geostrophic in upper SBL and RL.
- Overshoots geostrophic slightly, leading to super-geostrophic wind in SBL and RL.
- Nocturnal wind maximum results in upper SBL and RL ("low-level" jet, LLJ).
- Time period over which this occurs around 9 hours (although depends on latitude)
- Wind max occurs late night/early morning hours (3 to 6am-ish).

Mathematical illustration of above process shown in white-board notes …
See also handouts in class for typical wind speed profiles showing LLJ

Some amplifying effects in Southern U.S. Great Plains

(Kansas, Oklahoma)

- Gulf High Pressure forces southerly flow in area (southerly geostrophic wind)
- Sloping terrain upwards towards Rockies provides amplification of southerly geostrophic wind (as slope cooling occurs at night).
- Lee-side low development at times east of Rockies also amplifies southerly geostrophic wind.
- Result: Southern U.S. Great Plains very (!) conducive to LLJ development.