Assignment 3

April 11, 2014

Due by April 25th.

The first two assignments were worth 10 points each. This assignment is worth 20 points. Make sure you name your .tex file and the .pdf file with your full name. Like this: `DavidGoulette.tex`

I have intentionally crafted this assignment so that there are a few challenges you will have to figure out (nothing too bad though). But if you get stuck on something, please ask a question on piazza or email me and I will answer on piazza. Make sure you read the following instructions carefully.

I want you to recreate what I have written below, in sections 1 through 4 verbatim. So I want you to copy every detail exactly as I have it here to the best of your ability. EXCEPT, you do not have to copy any obvious side comments that I have placed in square brackets: ["blah blah comments"]. (There is one place where I have a closed *interval* $[a, b]$ and of course, that is not a comment.) Any comments inside square brackets are reminders and hints etc. But everything that is NOT a comment in brackets you must copy as best you can.

Section 5 is a free write section that I explain in section 5 below.

Here are some things you will need for this assignment:

`$\times$`	$\times$	
`$\cdot$`	$\cdot$	⇐ Small multiply dot.
`$\mathbb{R}$`	$\mathbb{R}$	⇐ Math Blackboard Script
`$\cap$`	$\cap$	⇐ Set intersection
`$\{\}$`	$\{\}$	⇐ Curly braces for sets
`$x\ \text{and}\ y$`	$x\ \text{and}\ y$	⇐ For text inside math with space
`$x\in S$`	$x \in S$	⇐ Meaning "x is in the set S":
`A \LaTeX\ example.`	A LaTeX example.	⇐ Fancy logo with proper spacing.
`A \LaTeX example.`	A LaTeXexample.	⇐ Bad spacing (don't do this).
`$\to$`	$\to$	⇐ Small right arrow
`\[\lim_{x\to a}f(x)\]`	$\lim_{x\to a} f(x)$	⇐ Limits.
`$\overline{z+w}$`	$\overline{z+w}$	⇐ Variable length over-bar.

Also, make sure you do in-line math symbols inside of dollar signs.
This is bad:

`Let f be a function of x.` $\Rightarrow$ Let f be a function of x.

This is correct:

`Let $f$ be a function of $x$.` $\Rightarrow$ Let f be a function of x.

Of course you will need to label things with the techniques that I laid out in the last section of the intro to LaTeX document. And you will need matrices as well.

Make sure you have added the AMS math packages as well as mathtools to your preamble. You may not need them all but you should eventually learn all of the things that are in these packages, so get in the habit of adding them.

```
\usepackage{amssymb}
\usepackage{amsthm}
\usepackage{amsmath}
\usepackage{mathtools}
```

Also add the following in your preamble so that your margins are the same and your links are obvious and live.

```
\usepackage[top=1in, bottom=1in, left=1in, right=1.5in]{geometry}
\usepackage[colorlinks=true,linkcolor=blue]{hyperref}
```

If you just manually type the labels then they will not be live links. You have to use proper labeling and referencing as outlined in the intro to LaTeX document.

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1 Labeling things and math practice

Here is an equation that I labeled:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1}$$

Equation (1) is the definition of the derivative for calculus on $\mathbb{R}$.

In section 2, we are going to practice doing matrices. Section 4 is on page 3. [Your page number will likely be different than mine because of my long intro with instructions. By-the-way, I didn't just type the numbers manually there, I used labels and referencing. So LaTeX did the numbering for me. I will check your code on this.]

Here is the fundamental theorem of Calculus. Let f be a continuous function defined on the closed interval $[a, b]$. We define a function F, for all x in $[a, b]$ to be the following:

$$F(x) = \int_a^x f(t)\, dt. \tag{2}$$

Then, F is continuous on $[a, b]$, it is differentiable on the open interval (a, b), and

$$F'(x) = f(x)$$

for all x in (a, b).

Equation (2) can be thought of as an area function since the value of $F(x)$ can be visualized as the signed area "under" the graph of f from a to x.

2 Matrices

Here is the general way you multiply a 2×2 matrix by a vector in $\mathbb{R}^2$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \tag{3}$$

Note that equation (3) is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T is represented by a matrix.

[The T in the last sentence is just a capital T in math mode.]

3 Text inside of math

[Note that the equations below are aligned so you need the `align` environment. Also the right hand side of those equations are text inside of math so you need to use the `\text{}` function.]

We define the following sets:

$$\begin{aligned} P &= \text{“The set of all people.”} \\ U &= \text{“People who were born in the United States.”} \\ S &= \text{“People who are at least 35 years old.”} \\ A &= \text{“People who have been a U.S. resident for 14 years.”} \end{aligned}$$

A person is not eligible to be the President of the United States of America if he or she is not in the following intersection of sets:

$$U \cap S \cap A = \{p \in P \mid p \in U \text{ and } p \in S \text{ and } p \in A\}$$

[In case you have never defined a set using set builder notation like this, the way you read this is: “The set includes each person p in the set of all people P, such that p is in U, and p is in S and p is in A. That is he or she was born in the U.S., is at least 35, and has been 14 years a resident. To get the word “and” inside the set above you need `\text{}`. Also, you need to force spacing so that the set definition is readable like a sentence. I used `\;` and also `\` followed by a space (see the section on horizontal spacing).]

4 Over lines and other various things

Every complex number, z, can be written $z = x + iy$, where x and y are real numbers, and i is the imaginary unit $i = \sqrt{-1}$. The ***real part*** of z is x, and the ***imaginary part*** of z is y. The complex conjugate of z is notated $\overline{z}$ and it is defined

$$\overline{z} = \overline{x + iy} = x - iy \tag{4}$$

So complex conjugation just changes the sign of the imaginary part of z.

The following identities are facts related to complex conjugation. Let z and w be complex numbers. Then

$$\overline{zw} = \overline{z} \cdot \overline{w} \tag{5}$$

$$\overline{z + w} = \overline{z} + \overline{w}. \tag{6}$$

Here we will prove (5). Let $z = x + iy$ and $w = u + iv$. Then

$$\begin{aligned}
\overline{zw} &= \overline{(x+iy)(u+iv)} \\
&= \overline{xu - yv + iyu + ixv} \\
&= \overline{(xu - yv) + i(yu + xv)} \\
&= (xu - yv) - i(yu + xv) \qquad (7) \\
&= xu - ixv - iyu - yv \\
&= x(u - iv) - iy(u - iv) \\
&= (x - iy)(u - iv) \\
&= \overline{(x+iy)} \cdot \overline{(u+iv)} \qquad (8) \\
&= \overline{z} \cdot \overline{w}.
\end{aligned}$$

Steps (7) and (8) are justified by the definition of conjugation: equation (4). Thus we have proven (5). Equation (6) can be proven in a similar manner.

[MAKE SURE YOU SEE THE LAST SECTION OF THE ASSIGNMENT ON THE NEXT PAGE!!]

5 Free choice

O.k., don't copy what I write in this section. Make sure this section is on a new page like this. In this last section I want you to write up *at least* one page of something mathematical (and that of course includes statistics if that is your thing, just make sure there are formulas and equations). What I would prefer is that you write up an explanation of something you are studying currently. Or it could be something you studied recently. So you could explain how to solve a certain type of problem, or you could explain a particular mathematical concept and have an example, or you could prove a theorem and provide an example etc. Feel free to do physics if that is your interest, as long as it has math in it. If you can't think of something to do, then just pull a calculus book off of the shelf and solve a couple of non-trivial problems (like solving a challenging integral with some tricky substitution, or integration by parts, or a triple integral in spherical coordinates or finding the radius of convergence of a Taylor series, or finding the centroid of a region via integration, etc.). In this section I don't want you to completely copy something from a book, I want you to write your own explanations, solutions or proofs as best you can (of course you can copy the statement of a problem or exercise from a book, but the solution should be yours). If you are currently taking a class and you are working on the homework, just type up your work and submit that to me.

Do your write-up as if you are trying to clearly explain what you are doing to a classmate at your academic level. So it could be like you are helping a classmate understand some concept that you are working on. So show your steps, and/or try to provide some explanation. Just explain whatever math is at your current level of understanding.

If you have any questions or are unsure about what to do, feel free to email me and ask a question.

Here is an example of what I mean. Don't copy my example.

This is something that you might not have thought about before:

You all know that the equation of a line in the two dimensional x, y plane, $\mathbb{R}^2$, is of the form $y = mx + b$. But have you ever thought about what the equation of a line would be using complex numbers? In order to answer this question we need to recall a few things. You might remember that the set of all complex numbers, $\mathbb{C}$, can be thought of as a plane of numbers. This is made obvious when you remember that every complex number can be written $z = x + iy$, which can be though of as a point (x, y), where x is the real part and y is the imaginary part of z. We often use the following notation for the real and imaginary part: $\text{Re}(z) = x$ and $\text{Im}(z) = y$. (In fact, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ and we can *define* a complex number as an ordered pair (x, y).)

Now in order to derive the complex form of a line we need the following identities for the real and imaginary parts of a complex number. Let $z = x + iy$. Then

$$\text{Re}(z) = x = \frac{z + \overline{z}}{2} \tag{9}$$

$$\text{Im}(z) = y = \frac{z - \overline{z}}{2i}. \tag{10}$$

Here is a proof of (10):

$$\begin{aligned}\frac{z-\overline{z}}{2i} &= \frac{(x+iy)-\overline{(x+iy)}}{2i}\\ &= \frac{(x+iy)-(x-iy)}{2i}\\ &= \frac{2iy}{2i}\\ &= y.\end{aligned}$$

Equation (9) is similarly proven.

Now, to derive the complex form of a line, we simply need to take, $y = mx + b$ and substitute the right hand side of (9) and (10) for x and y:

$$\begin{aligned} y &= mx+b\\ \frac{z-\overline{z}}{2i} &= m\left(\frac{z+\overline{z}}{2}\right)+b\\ z-\overline{z} &= im(z+\overline{z})+2ib.\end{aligned}$$

Rearranging terms gives us:

$$(1-im)z-(1+im)\overline{z}-2ib=0.$$

Multiplying through by $i/2$ gives us:

$$\begin{aligned}\left(\frac{m+i}{2}\right)z+\left(\frac{m-i}{2}\right)\overline{z}+b=0.\\ \beta z-\overline{\beta}\,\overline{z}+\gamma=0, \qquad (11)\end{aligned}$$

where, in the last step, we have relabeled $\beta = (m+i)/2$ and $\gamma = b$. So equation (11) is the complex equation of a line in the complex plane. In other words, if you choose a particular $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, then a Euclidean line in the complex plane is the following set points:

$$\{z \in \mathbb{C} \mid \beta z-\overline{\beta}\,\overline{z}+\gamma=0\}.$$